HARMONIC Q.C. SELF-MAPPING AND MÖBIUS TRANSFORMATIONS OF THE UNIT BALL B^n

DAVID KALAJ AND MIODRAG MATELJEVIĆ

Radim na tekstu; very rough version

Proposition 0.1. [16] *Let* $u : B^n \to \Omega$, $n \geq 3$ *be twice differentiable q.c. mapping of the unit ball onto the bounded domain* Ω *with* C^2 *boundary satisfying the differential inequality:*

$$|\Delta u| \leq A|\nabla u|^2 + B,\ A, B \geq 0.$$

Then ∇u *is bounded and* u *is Lipschitz continuous.*

Ovu prop nisam dokazivao;izgleda interesantno;U pristupu u ovom radu koji se verovatno moze uopstiti lopta ima vaznu ulogu? O kojem radu mislite?????

1. co-Lipschitz continuity

It is well-known that quasiconformal maps are locally well-behaved with respect to distance distortion. If $f : \Omega \mapsto \Omega'$ is a K -quasiconformal mapping between domains $\Omega, \Omega' \subset \mathbb{R}^n$, then f is locally Hölder continuous with exponent $\alpha = K^{1/(1-n)}$, i.e.

$$|f(x) - f(y)| \leq M|x - y|^\alpha \tag{1.1}$$

whenever x and y lie in a fixed compact set E in Ω. Here M is a constant depending only on K and E which can in general tend to infinity as the distance, from E to the boundary of Ω tends to zero. However if the boundary of Ω is enough "regular", then there hold an inequality similar to (1.1) uniformly in Ω (see [10]). The following lemma in some form is proved in [19]. For the completeness we give its proof here and show that the constant is sharp.

Lemma 1.1. *If* $u \in C^{1,1}$ *is a* $K-$ *quasiconformal mapping, defined in a domain* $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$, *then*

$$J_u(x) > 0, x \in \Omega$$

providing that $K < 2^{n-1}$. *The constant* 2^{n-1} *is sharp.*

Proof. Assume converse, i.e. $J_u(a) = 0$ for some $a \in \Omega$. This implies $\nabla u(a) = 0$. Without loos of generality we can assume that, $a = 0$ and $u(0) = 0$. Let $r < \text{dist}(0, \partial\Omega)$ and take $E = B(0, r)$. Applying (1.1) to the mapping, $f = u^{-1}$, defined in $\Omega' = u(\Omega)$, we obtain

$$|f(y)| \leq M_E|y|^{K^{1/(1-n)}}, \text{for } y \in u(E). \tag{1.2}$$

2000 *Mathematics Subject Classification.* Primary 30C65; Secondary 31B05.
Key words and phrases. Quasiconformal maps, PDE, Lipschitz condition.

This implies

$$M_E^{-K^{1/(n-1)}}|x|^{K^{1/(n-1)}} \le |u(x)|, \text{for } x \in E. \tag{1.3}$$

Now since u is twice differentiable, with $\nabla u(0) = 0$ and $u(0) = 0$, by Taylor formula, it follows that there exists a positive constant N such that

$$|u(x)| \le N|x|^2, x \in E. \tag{1.4}$$

Combining (1.3) and (1.4) it follows that

$$M_E^{-K^{1/(n-1)}}/N \le |x|^{2-K^{1/(n-1)}}, x \in E. \tag{1.5}$$

This is only possible providing that

$$2 - K^{1/(n-1)} \le 0.$$

And thus $K \ge 2^{n-1}$ which is a contradiction.

To prove the sharpness of the result, take the mapping $u(x) = |x|^\alpha x$, with $\alpha \ge 1$. Then

$$J_u(x) = (1+\alpha)|x|^{n\alpha}, \tag{1.6}$$

and

$$|\nabla u(x)| = (\alpha+1)|x|^\alpha. \tag{1.7}$$

By (1.6) and (1.7) it follows that

$$\frac{|\nabla u(x)|^n}{J_u(x)} = (\alpha+1)^{n-1}.$$

Therefore u is twice differentiable $(1+\alpha)^{n-1}$-quasiconformal self-mapping of the unit ball with $J_u(0) = 0$. This means that the constant 2^{n-1} is the best possible. □

Lemma 1.2. *Let u be a harmonic mapping of the unit ball into itself and let $u(0) = 0$. Then there exists a constant C_n such that*

$$\frac{1-|x|^2}{1-|u(x)|^2}| \le C_n, x \in B^n. \tag{1.8}$$

Proof. Let S^+ denotes the northern hemisphere and let S^- denotes the southern hemisphere. Let $U = P[\chi_{S^+}] - P[\chi_{S^-}]$ be the poisson integral of a function that equals 1 on S^+ and -1 on S^-. Then by Schwartz lemma ([1]), for fixed x_0 there holds the inequality

$$\left\langle u(x), \frac{u(x_0)}{|u(x_0)|} \right\rangle \le |U(|x|N)|,$$

where N is the north pole.

It follows that

$$|u(x_0)|^2 \le |U(|x_0|N)|^2.$$

Thus

$$\frac{1-|x|^2}{1-|u(x)|^2} \le \frac{1-|x|^2}{1-U(|x|N)^2} =: g(r), \quad r = |x|.$$

Now we need the following lemma

Lemma 1.3 (Hopf's Boundary Point lemma)**.** [18] *and* [5]. *Let v satisfies $\Delta v \geq 0$ in D and $v \leq M$ in D, $v(P) = M$ for some $P \in \partial D$. Assume that P lies on the boundary of a ball $B \subset D$. If v is continuous on $D \cup P$ and if the outward directional derivative $\frac{\partial v}{\partial n}$ exists at P, then $v \equiv M$ or*

$$\frac{\partial v}{\partial n} > 0.$$

Let apply this lemma to the function $U(x)$ and take $h(r) = U(rN)$. We obtain

$$h'(1) = \frac{\partial U(N)}{\partial n} > 0.$$

Thus

$$C_n := \sup_{|x|\leq 1} \left\{ \frac{1-|x|^2}{1-U(|x|N)^2} \right\} < \infty.$$

The results follows.

□

Lemma 1.4. *If p is a Möbius transformation of the unit ball onto itself, then for every $k, l \in \mathbb{N}$ there exist constants $C_{k,l}$ such that*

$$\frac{k!|a|^{k-1}(1-|a|^2)}{[x,a]^{k+1}} \leq |p^{(k)}(x)| \leq \frac{C_{k,0}|a|^{k-1}(1-|a|^2)}{[x,a]^{k+1}},\ x \in B^n,\ p(0) = a. \tag{1.9}$$

$$\frac{|p^{(k)}(x)|}{|p^{(l)}(x)|} \leq C_{k,l} \frac{1}{(1-|x|)^{k-l}} \quad x \in B^n. \tag{1.10}$$

and

$$|p^{(k)}(x)| \leq \frac{C_{k,0}}{(1-|p(0)|^2)^{(k-1)/2}} \left(\frac{1-|p(x)|^2}{1-|x|^2} \right)^{(k+1)/2} \qquad x \in B^n. \tag{1.11}$$

Proof. Since

$$|\nabla p| = \frac{1-|a|^2}{[x,a]^2},$$

using the fact that

$$|\nabla |\nabla p|| \leq |\nabla \nabla p|,$$

it follows that

$$\frac{2|a|(1-|a|^2)}{[x,a]^3} \leq |\nabla \nabla p|.$$

The rest of the proof of left-hand side of (1.9) follows by induction.

Some of the formula which we use here are taken form the excellent book of Ahlfors [2]. From

$$\begin{aligned}[x,a]^2 &= |x|^2|x^* - a|^2 = |a|^2|x - a^*|^2 \\ &= ||x|a - |x|x^*|^2 = ||a|x - |a|a^*|^2 = ||x|a - |x|x^*| \cdot ||a|x - |a|a^*|,\end{aligned}$$

it follows that

$$[x,a]^2 \geq (1-|a|)(1-|x|).$$

Assume that p is not an identity and let $p(0) = -a$ or $p(a) = 0$. Then

$$p(x) = \frac{(1-|a|^2)(x-a) - |x-a|^2 a}{[x,a]^2},$$

and

$$P'(x) = \frac{1-|a|^2}{[x,a]^2}\Delta(x,a),$$

where

$$\Delta(x,a) = (I - 2Q(a))(I - 2Q(x-a^*)),$$

and $Q(y)$ is the matrix which components have the form

$$Q(y)_{i,j} = \frac{y_i y_j}{|y|^2}.$$

For every y there holds $K(y) := I - 2Q(y) \in O_n$ (is an orthogonal matrix). Thus $\Delta(x,a)$ is an orthogonal matrix as well and consequently

$$|\Delta(x,a)| = 1.$$

This means that

$$|p'(x)| = \frac{1-|a|^2}{[x,a]}.$$

Thus

$$|p'(x)| \leq \frac{2}{1-|x|}. \tag{1.12}$$

If we put

$$A = \frac{1-|a|^2}{[x,a]^2}$$

and

$$B = (I - 2Q(a))(I - 2Q(x-a^*)),$$

then we have that

$$p' = AB,$$

and consequently for $k+1-$ derivative of p we have

$$p^{(k+1)}(x) = \sum_{j=1}^{k} \binom{k}{j} A^{(j)} B^{(k-j)}, \tag{1.13}$$

treated as a k linear form between $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$ and $M_{n\times n}$.

We will use the following notation

$$Q(y) = \frac{y \otimes y}{|y|^2},$$

where $\otimes$ denotes the tensor product of vectors.

Differentiating we obtain

$$
\begin{aligned}
Q'(y)h_1 &= \frac{h_1 \otimes y + y \otimes h_1}{|y|^2} - 2\frac{\langle h_1, y\rangle\, y \otimes y}{|y|^4} \\
&= \frac{|y|^2(h_1 \otimes y + y \otimes h_1) - 2\,\langle h_1, y\rangle\, y \otimes y}{|y|^4} \\
&= \frac{P^1(y, y, y, h_1)}{|y|^4}.
\end{aligned} \tag{1.14}
$$

Let us prove that, for $k \in \mathbb{N}$, there exists a $2k+2$ linear form

$$P^k : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \mapsto M_{n\times n}$$

such that

$$Q^{(k)}(y)(h_1, h_2, \dots h_k) = \frac{1}{|y|^{2k+2}} P^k(y, \dots, y, h_1, \dots, h_k). \tag{1.15}$$

To prove this, we use the induction.

It is evident that, according to (1.14), this is true for $k = 1$.

Assume that (1.15) is true for k and prove it for $k+1$. By (1.15) it follows that

$$
\begin{aligned}
Q^{(k+1)}(y)(h_1, &h_2, \dots h_k, h_{k+1}) \\
&= \frac{1}{|y|^{2k+2}} \sum_{j=1}^{k+2} P^k(y, \dots, y\ \overset{j\downarrow}{h_{k+1}}, y \dots, y, h_1, \dots, h_k) \\
&- (k+1)\frac{\langle y, h_{k+1}\rangle}{|y|^{2k+4}} P^k(y, \dots, y, h_1, \dots, h_k),
\end{aligned} \tag{1.16}
$$

where $\overset{j\downarrow}{h_{k+1}}$ denotes that h_{k+1} is in j^{th} position.

Thus

$$Q^{(k+1)}(y)(h_1, h_2, \dots h_k, h_{k+1}) = \frac{P^{k+1}(y, \dots, y, h_1, \dots, h_k, h_{k+1})}{|y|^{2(k+1)+2}}, \tag{1.17}$$

where

$$
\begin{aligned}
P^{k+1}&(e_1, \dots, e_{k+3}, f_1, \dots f_{k+1}) \\
&= \sum_{j=1}^{k+1} < e_{k+3}, e_j > P^k(e_1, \dots, \overset{j\downarrow}{f_{k+1}}, \dots, e_{k+2}, f_1, \dots, f_k) \\
&- (k+1) < e_{k+3}, f_{k+2} > P^k(e_1, \dots, e_{k+2}, f_1, \dots, f_k).
\end{aligned}
$$

Since P^k is an $2k+2$ linear form, it follows that

$$|P^k(y, \dots, y, h_1, \dots, h_k)| \le |P^k||y|^{k+2} \prod_{j=1}^{k} |h_j|. \tag{1.18}$$

Thus

$$|Q^k(y)| \le \frac{|P^k|}{|y|^k}, \tag{1.19}$$

or what is the same

$$|Q^k(x-a^*)| \leq \frac{|P^k|}{|x-a^*|^k} = \frac{|a|^k|P^k|}{[x,a]^k}. \tag{1.20}$$

To continue, observe that

$$B(x) = K(a)(I - 2Q(x-a^*)),\ K(a) \in O_n.$$

Thus

$$B^{(k)}(x) = -2K(a)Q^{(k)}(x-a^*),$$

and using the identity

$$\frac{1-|p(x)|^2}{1-|x|^2} = \frac{1-|a|^2}{[x,a]^2} = \frac{1-|a|^2}{|a|^2|x-a^*|^2},$$

we obtain

$$|B^k(x)| \leq \frac{|a|^k|P^k|}{[x,a]^k} < \frac{2|a|^k|P^k|(1-|p|^2)^{k/2}}{(1-|a|^2)^{k/2}(1-|x|^2)^{k/2}}. \tag{1.21}$$

In order to estimate the derivatives of $A(x) = \frac{1-|a|^2}{[x,a]^2}$, define

$$H(y) = \frac{1}{|y|^2} = \frac{|a|^2}{1-|a|^2}A(x),\quad y = x-a^*.$$

Similarly as above it can be proved that, for every $k \geq 1$ there exists a $2k$ linear form

$$G^k(x) : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \mapsto \mathbb{R}$$

such that

$$H^{(k)}(y)(h_1, h_2, \ldots h_k) = \frac{1}{|y|^{2k+2}}G^k(y, \ldots, y, h_1, \ldots, h_k). \tag{1.22}$$

Therefore

$$|H^k(y)| \leq \frac{|G^k|}{|y|^{k+2}}, \tag{1.23}$$

and having in mind

$$\frac{1-|p(x)|^2}{1-|x|^2} = \frac{1-|a|^2}{[x,a]^2},$$

it follows

$$\begin{aligned} |A^k(x)| &\leq \frac{(1-|a|^2)|G^k|}{|a|^2|x-a^*|^{k+2}} = \frac{|a|^k(1-|a|^2)|G^k|}{[x,a]^{k+2}} \\ &\leq \frac{2|a|^k|G^k|(1-|p|^2)^{1+k/2}}{(1-|a|^2)^{(k-1)/2}(1-|x|^2)^{1+k/2}}. \end{aligned} \tag{1.24}$$

Combining (1.13), (1.21) and (1.24) we obtain for $k \geq 1$:

$$|p^{(k+1)}(x)| \leq C_{k+1}\frac{1}{[x,a]^{k+1}}, \tag{1.25}$$

and

$$|p^{(k+1)}(x)| \leq C_{k+1}\frac{1}{(1-|x|)^{k+1}}, \tag{1.26}$$

where

$$C_{k+1} = 2\sum_{j=1}^{k} \binom{k}{j} |P^j||G^{(k-j)}|.$$

This completes the proof. □

Remark 1.5. In the plane case the sharp constant in (1.26) is $C_k = 2k!$. The question arises, is this the best constant for arbitrary n? Notice that in (1.12) is showed that this is the case for $k = 1$.

Theorem 1.6. *If u is a q.c. harmonic mapping of the unit ball onto itself with $K < 2^{n-1}$ then*

$$J_u(x) \geq c_K > 0 \ \text{ for } x \in B^n.$$

Proof. We will use similar approach as in [19]. We will prove that the function $|\nabla u|$ is uniformly bounded below away from 0 by contradiction. Suppose not, then there exists a sequence of points $x_i \in B^n$, such that $\nabla u(x_i) \to 0$ as $i \to \infty$. To finish the proof we prove the following lemma:

Lemma 1.7. *Let u be a harmonic Lipschitz mapping of the unit ball into itself. Let x_i be a sequence of points. Let p_i and q_i be two Möbius transformations of B^n such that $q_i(0) = x_i$ and $p_i(u(x_i)) = 0$. Take $u_i = p_i \circ u \circ q_i$. Then*

$$|D^{(k)} u_i(x)| \leq C_n^k \frac{1}{(1-|x|^2)^k}, \ \ k \in \mathbb{N}, \tag{1.27}$$

where C_n^k is independent on x and i.

Proof. In order to simplify calculations, sometimes along this proof, we will avoid the arguments of functions.

Using

$$|\nabla p_i(u)| = \frac{1-|p_i(u)|^2}{1-|u|^2}, \tag{1.28}$$

and

$$|\nabla q_i(x)| = \frac{1-|q_i(x)|^2}{1-|x|^2} \tag{1.29}$$

it follows that

$$\begin{aligned} |\nabla u_i| &\leq |\nabla p_i||\nabla u||\nabla q_i| \\ &\leq \frac{1-|p_i(u(q_i(x)))|^2}{1-|u(q_i(x))|^2}\frac{1-|q_i(x)|^2}{1-|x|^2}|\nabla u| \\ &\leq C_n|\nabla u|_\infty \frac{1-|p_i(u(q_i(x)))|^2}{1-|x|^2}. \end{aligned} \tag{1.30}$$

Thus

$$|\nabla u_i| \leq C_n |\nabla u|_\infty \frac{1}{1-|x|^2}. \tag{1.31}$$

For $m \in \mathbb{N}$, we make use of the Cauchy inequalities:

$$|D^m(u)(q_i(x))| \leq A_n \frac{|\nabla u|_\infty}{(1-|q_i(x)|)^{m-1}}. \tag{1.32}$$

To establish the behaviors of $D^k u_i$, $k>1$, we use induction. It is evident that, $D^k u_i$ is complicate to compute for large k, however it is clare that it can be written as the following sum:

$$D^k u_i = \sum \left(p_i^{(\tau)} \prod_{t=1}^{n} D^{j_t} u q_i^{(s_{t1})} \cdot \dots \cdot q_i^{(s_{tl_t})} \right), \tag{1.33}$$

where $\prod$ denotes corresponding product of linear operators. Here the index τ ranges from 1 to k, and the other indexes j_t,s_{t1}, …, s_{tl_t} satisfy similar bounds.

Since

$$|p_i^{(\tau)}| \prod_{t=1}^{n} |D^{j_t} u||q_i^{(s_{t1})}| \cdot \dots \cdot |q_i^{(s_{tl_t})})| \le C|p_i^{(\tau)}| \prod_{t=1}^{n} \frac{|\nabla u|_\infty}{(1-|q_i(x)|)^{j_t-1}} |q_i^{(s_{t1})}| \cdot \dots \cdot |q_i^{(s_{tl_t})})|$$

It is enough to prove that

$$|p_i^{(\tau)}| \prod_{t=1}^{n} \frac{|\nabla u|_\infty}{(1-|q_i(x)|)^{j_t-1}} |q_i^{(s_{t1})}| \cdot \dots \cdot |q_i^{(s_{tl_t})})| \le C \frac{1}{(1-|x|)^k}. \tag{1.34}$$

For $k=1$, (1.34) is true. Assume that (1.34) is true for k an therefore (1.33) is true as well. In what follows we are going to prove (1.34) for $k+1$.

Since $D^{k+1}u_i = DD^k u_i$, it follows that, a corresponding formula (1.33) for $D^{k+1}u_i$ instead of

$$p_i^{(\tau)} D^{j_t} u q_i^{(s_{t1})} \cdot \dots \cdot q_i^{(s_{tl_t})}$$

contains

$$\Big(p_i^{(\tau+1)} Du q_i' D^{j_t} u q_i^{(s_{t1})} \cdot \dots \cdot q_i^{(s_{tl_t})}) + p_i^{(\tau)} D^{j_t+1} u q_i' \cdot q_i^{(s_{t1})} \cdot \dots \cdot q_i^{(s_{tl_t})} +$$
$$p_i^{(\tau)} D^{j_t} u q_i^{(s_{t1+1})} \cdot \dots \cdot q_i^{(s_{tl_t})}) + \dots + p_i^{(\tau)} D^{j_t} u q_i^{(s_{t1})} \cdot \dots \cdot q_i^{(s_{tl_t+1})}) \Big),$$

and consequently the corresponding formula (1.34), instead of

$$|p_i^{(\tau)}| \frac{|\nabla u|_\infty}{(1-|q_i(x)|)^{j_t-1}} |q_i^{(s_{t1})}| \cdot \dots \cdot |q_i^{(s_{tl_t})}|$$

contains

$$\Big(|p_i^{(\tau+1)}||Du||q_i'||D^{j_t}u||q_i^{(s_{t1})}| \cdots |q_i^{(s_{tl_t})})| + |p_i^{(\tau)}| \frac{|\nabla u|_\infty}{(1-|q_i(x)|)^{j_t}} |q_i'| \cdot |q_i^{(s_{t1})}| \cdots |q_i^{(s_{tl_t})}|$$
$$+ |p_i^{(\tau)}| \frac{|\nabla u|_\infty}{(1-|q_i(x)|)^{j_t-1}} ||q_i^{(s_{t1+1})}| \dots |q_i^{(s_{tl_t})}|) + \dots$$
$$+ |p_i^{(\tau)}| \frac{|\nabla u|_\infty}{(1-|q_i(x)|)^{j_t-1}} |q_i^{(s_{t1})}| \cdots |q_i^{(s_{tl_t+1})})| \Big).$$

Applying now (1.29), we get

$$\begin{aligned} \frac{|\nabla u|_\infty}{(1-|q_i(x)|)^{j_t}} |q_i'| &= \frac{|\nabla u|_\infty}{(1-|q_i(x)|)^{j_t}} \frac{1-|q_i(x)|^2}{1-|x|^2} \\ &\le \frac{|\nabla u|_\infty}{(1-|q_i(x)|)^{j_t-1}} \frac{2}{1-|x|}. \end{aligned} \tag{1.35}$$

Next, by applying (1.8) and (1.10) we obtain

$$|p_i^{(\tau+1)}Duq_i'|/|p_i^{(\tau)}| \le \frac{C|p_i^{(\tau)}|}{1-|u(q_i(x))|}\frac{1-|q_i(x)|}{1-|x|} \le C\frac{|p_i^{(\tau)}|}{1-|x|}. \tag{1.36}$$

On the other hand, according to (1.10) we have

$$|q_i^{(j+1)}| \le C\frac{|q_i^{(j)}|}{1-|x|}. \tag{1.37}$$

By induction, (1.34) is true for k. The last fact and equation (1.35), (1.36) and (1.37) imply that (1.34) is true for $k+1$. Consequently

$$|D^{(k)}u_i(x)| \le C_n^k\frac{1}{(1-|x|^2)^k}, \quad k \in \mathbb{N} \tag{1.38}$$

□

Taking the notations of the previous lemma, $u_i = p_i\circ u\circ q_i$ is a C^∞ $K-$quasiconformal mapping of the unit ball onto itself satisfying the condition $u_i(0) = 0$ and

$$|\nabla u_i(0)| = \frac{1-|x_i|^2}{1-|u(x_i)|^2}|\nabla u(x_i)| \to 0 \tag{1.39}$$

as $i \to \infty$. By [6] for example, a subsequence of u_i, also denoted by u_i, converges uniformly to a K-quasiconformal map u on the close unit ball B^n. According to this lemma, u is in $C^\infty(B^n; B^n)$ with $u(0) = 0$ and from (1.39) $\nabla(u)(0) = 0$. This obviously contradicts the statement of Lemma 1.1. Hence the proof of the proposition is completed.

□

Remark 1.8. Using the formula

$$|\nabla q_i(x)| = \frac{1-|x_i|^2}{|x_i||x+x_i^*|}, \tag{1.40}$$

First of all we have

$$1-|p_i(u(q_i(x)))|^2 = \frac{(1-|u(x_i)|^2)(1-|u(q_i(x))|^2)}{|u(x_i)|^2|u(q_i(x))-u(x_i)^*|^2}, \tag{1.41}$$

Next there holds

$$1-|u(q_i(x))|^2 \le |\nabla u|_\infty\frac{1+|u(q_i(x))|}{1+|q_i(x)|}(1-|q_i(x)|^2) \le 2|\nabla u|_\infty(1-|q_i(x)|^2), \tag{1.42}$$

$$(1-|q_i(x)|^2) = |\nabla q_i|(1-|x|^2), \tag{1.43}$$

and

$$|u(x_i)|^2|u(q_i(x))-u(x_i)^*|^2 \ge (1-|u(x_i)|)^2. \tag{1.44}$$

Using one more again (1.8) and combining (1.41), (1.42), (1.43) and (1.40) it follows that

$$\frac{1-|p_i(u(q_i(x)))|^2}{1-|x|^2} \le 2|\nabla u|_\infty \frac{(1+|u(x_i)|)^2}{|x_i||x-x_i^*|}\frac{1-|x_i|^2}{1-|u(x_i)|^2} \le \frac{8C_n|\nabla u|_\infty}{|x_i||x+x_i^*|}. \tag{1.45}$$

Assume that $\lim_{i\to\infty} x_i = t$. Combining (1.30) and (1.45) we obtain that for $x \in B^n \setminus \{x : |x+t| \le \varepsilon\}$.

$$|\nabla u_i| \le \frac{16}{\varepsilon} C_n^2 |\nabla u|_\infty^2, i \ge i_0.$$

It follows that u_n is locally uniformly Lipschitz family of q.c. mappings with locally bounded derivative.

Theorem 1.9. *Let $K < 2^{n-1}$ and assume that u is a $K-$q.c. harmonic mapping the unit ball onto itself. Then it is a bi-Lipschitz mapping.*

Proof. First of all

$$1/c < |\nabla u| \le c,$$

then using the fact u is quasi-conformal, it follows that

$$1/c_1 < |\nabla(u^{-1})| \le c_1$$

and this implies that u is bi-Lipschitz.

□

In what follows is given an non-trivial example of q.c. harmonic selfmapping of the unit ball.

Example 1.10. Let $I_\varepsilon(x) = (x_1 + \varepsilon, x_2, x_3)$ then

$$|I_\varepsilon(x)| = (1 + 2\varepsilon x_1 + \varepsilon^2)^{1/2}.$$

Define

$$\phi_\varepsilon(x) = I_\varepsilon(x)/|I_\varepsilon(x)| = (1 + 2\varepsilon x_1 + \varepsilon^2)^{-1/2}(x_1 + \varepsilon, x_2, x_3)$$

and take $\Phi_\varepsilon = P[\phi_\varepsilon]$. Then for small enough ε Φ_ε is a diffeomorphism of the unit ball onto itself having a diffeomorphic extension to the boundary. This for example means that Φ_ε is q.c.

It is enough to prove that Φ_ε is injective in $\overline{B^3}$ for small ε. In order to do this we use the following result due to Gilbarg and Hörmander see [7, Theorem 6.1 and Lemma 2.1],

Proposition 1.11. *The Dirichlet problem $\Delta u = f$ in Ω, $u = u_0$ on $\partial\Omega \in C^1$ has a unique solution $u \in C^{1,\alpha}$, for every $f \in C^{0,\alpha}$, and $u_0 \in C^{1,\alpha}$, and we have*

$$||u||_{1,\alpha} \le C(||u_0||_{1,\alpha,\partial\Omega} + ||f||_{0,\alpha}) \tag{1.46}$$

where C is a constant.

Direct calculations yield

$$|\nabla\Phi_\varepsilon(x)-Id| = |\nabla P[\phi_\varepsilon - Id](x)|$$

$$\le C\sup_{|x|=1}\{(\sum_{i=1}^{3}|\partial_{x_i}(\nabla\phi_\varepsilon(x)-Id)|^2)^{1/2}+(\sum_{i=1,j=1}^{3}|\partial_{x_i,x_j}(\nabla\phi_\varepsilon(x)-Id)|^2)^{1/2}\}$$

$$= C\times\sup_{|x|=1}\left\{\left(\left(-1+\frac{\varepsilon x_1+1}{(1+\varepsilon^2+2\varepsilon x_1)^{3/2}}\right)^2+2\left(-1+\frac{1}{\sqrt{1+\varepsilon^2+2\varepsilon x_1}}\right)^2\right.\right.$$

$$\left.+\frac{\varepsilon^2x_2^2}{(1+\varepsilon^2+2\varepsilon x_1)^3}-\frac{\varepsilon x_3}{(1+\varepsilon^2+2\varepsilon x_1)^{3/2}}\right)^{1/2}+\left(2(-1+\frac{1}{\sqrt{1+a^2+2ax_1}})^2\right.$$

$$+(-1-\frac{a(a+x_1)}{1+a^2+2ax_1{}^{3/2}}+\frac{1}{\sqrt{1+a^2+2ax_1}})^2$$

$$\left.\left.+\frac{a^2x_2^2}{(1+a^2+2ax_1)^3}+\frac{a^2x_3^2}{(1+a^2+2ax_1)^3}\right)^{1/2}\right\}.$$

Therefore

$$\lim_{\varepsilon\to 0}|\nabla\Phi_\varepsilon(x)-Id| = 0$$

uniformly on B^n.

It follows that there exist $\varepsilon > 0$ such that

$$\sup_{|x|\le 1}|\nabla\Phi_\varepsilon(x)-Id| < 1/2.$$

From

$$|\Phi_\varepsilon(x)-\Phi_\varepsilon(y)+y-x| \le 1/2|x-y|,$$

we obtain

$$1/2|x-y| \le |\Phi_\varepsilon(x)-\Phi_\varepsilon(y)|.$$

This implies that, Φ_ε is injective.

Remark 1.12. It seems that, the previous example can be modified to the class of all bi-Lipschitz harmonic diffeomorphism of the unit ball onto itself. Thus small perturbations of the boundary value of harmonic q.c. transformation $\phi \in C^2(\overline{B^n})$, of the unit ball onto itself induce harmonic q.c. mappings.

References

[1] Sheldon Axler, Paul Bourdon, Wade Ramey: *Harmonic function theory*, Springer Verlag, New York, 1992.

[2] L. Ahlfors: *Möbius transformations in several dimensions.* Ordway Professorship Lectures in Mathematics. University of Minnesota, School of Mathematics, Minneapolis, Minn., 1981. ii+150 pp.

[3] G.D. Anderson, M.K. Vamanamurthy: *Hölder continuity of quasiconformal mappings of the unit ball.* Proc. Am. Math. Soc. 104, No.1, 227-230 (1988).

[4] E. Heinz, *On one-to-one harmonic mappings*, Pacific J. Math. **9** (1959), 101–105.

[5] E. Hopf: *A remark on linear elliptic differential equations of second order*, Proc. Amer. Math. Soc., 3, 791-793 (1952).

[6] R. Fehlmann, M. Vuorinen: *Mori's theorem for n-dimensional quasiconformal mappings.* Ann. Acad. Sci. Fenn., Ser. A I, Math. 13, No.1, 111-124 (1988).

[7] D. Gilbarg, L. Hörmander: *Intermediate Schauder estimates.* Arch. Ration. Mech. Anal. 74, 297-318 (1980).

[8] D. Gilbarg, N. Trudinger: *Elliptic Partial Differential Equations of Second Order*, Vol. 224, 2 Edition, Springer 1977, 1983.
[9] J. Heinonen: *Lectures on Lipschitz Analysis,* Lectures at the 14th Jyvskyl Summer School in August 2004.
[10] Koskela, P.; Onninen, J.; Tyson, J. *Quasihyperbolic boundary conditions and capacity: Hölder continuity of quasiconformal mappings.* Comment. Math. Helv. 76 (2001), no. 3, 416–435.
[11] D. Kalaj: *Quasiconformal harmonic functions between convex domains*, Publ. Inst. Math., Nouv. Ser. 76(90), 3-20 (2004).
[12] D. Kalaj, M. Mateljević: *Inner estimate and quasiconformal harmonic maps between smooth domains*, Journal d'Analise Math. 100. (2006) 117-132.
[13] D. Kalaj; M. Pavlović: *Boundary correspondence under harmonic quasiconformal homeomorfisms of a half-plane* Ann. Acad. Sci. Fenn., Math. 30, No.1, (2005) 159-165.
[14] D. Kalaj; M. Pavlović: *On quasiconformal self-mappings of the unit disk satisfying the PDE* $\Delta u = g$, to appear.
[15] D. Kalaj: *Quasiconformal and harmonic mappings between Jordan domains,* Math. Z. Volume 260, Number 2, 237-252, 2008.
[16] D. Kalaj: *Interior estimate in the space* arXiv:0712.3580 (December 2007), 24 pages.
[17] D. Kalaj: *On harmonic quasiconformal self-mappings of the unit ball,* Ann. Acad. Sci. Fenn. Math. Vol 33, 261-271, (2008).
[18] M. H. Protter and H. F. Weinberger: *Maximum principles in differential equations,* Prentice Hall, Englewood Cliks N.J., 1967.
[19] Tam, Luen-Fai; Wan, Tom Y.-H. On quasiconformal harmonic maps. Pacific J. Math. 182 (1998), no. 2, 359–383.
[20] H. Lewy: *On the non-vanishing of the Jacobian in certain one-to-one mappings,* Bull. Amer. Math. Soc. 42. (1936), 689-692.
[21] O. Martio: *On harmonic quasiconformal mappings*, Ann. Acad. Sci. Fenn., Ser. A I 425 (1968), 3-10.
[22] I. S. Ovcinikov, G. D. Surovov: *Obobscennie proizvodnie i diferenciruemost pocti vsjudu* Mat. sbor., 75, N3, (1968), 323-334 (Russian).
[23] M. Pavlović: *Boundary correspondence under harmonic quasiconformal homeomorfisms of the unit disc*, Ann. Acad. Sci. Fenn., Vol 27, 2002, 365-372.
[24] V.S. Vladimirov: *Equations of mathematical physics* , MIR (1984) (Translated from Russian).
[25] Partyka, Dariusz, and Ken-ichi Sakan: *On bi-Lipschitz type inequalities for quasiconformal harmonic mappings,* Ann. Acad. Sci. Fenn. Math.. Vol 32, pp. 579-594 (2007).
[26] J. Resetnjak: *Ocenki modula neprerivnosti dlja nekotorih otobrazenij,* Sib. Mat. Jour., VII, N5 (1966), 1100 -1114.

University of Montenegro, Faculty of Natural Sciences and Mathematics, Cetinjski put b.b. 81000 Podgorica, Montenegro
E-mail address: davidk@cg.yu

University of Belgrade, Faculty of Mathematics, Studentski trg 16, 11000 Belgrade, Serbia
E-mail address: miodrag@matf.bg.ac.yu