ON THE UNIVALENT SOLUTION OF PDE $\Delta u = f$ BETWEEN SPHERICAL ANNULI

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Abstract. It is proved that if u is the solution of PDE $\Delta u = f$, that maps two annuli on the space $\mathbb{R}^3$, then the annulus in co-domain cannot be with arbitrary small modulus, providing that the annulus of domain is fixed. Also it is improved the inequality obtained in [2] for harmonic functions in $\mathbb{R}^3$. Finally it is given the new conjecture for harmonic mappings in the space similar to the conjecture of J. C. C. Nitsche for harmonic mapping in the plane related to the modulus of annuli.

1. Introduction and auxiliary results

Here $B(0,1)$ is the unit ball and $S(0,1)$ is the unit sphere, Ω is bounded homeomorphic image of the ball. We will consider two norms of $A = (a_{ij})_{i,j=1}^n$:

$$||A|| = \sup\{||Ax|| : ||x|| = 1\}$$

and

$$||A||_2 = \sqrt{\sum_{i,j=1}^n a_{ij}^2}.$$

We will also consider the function

$$\operatorname{trace} A = \sum_{i=1}^n a_{ii}.$$

If A is nonsingular matrix then there exists A^{-1} which is given by the formula:

$$A^{-1} = \frac{1}{\det A}(\tilde{A})^T,$$

where $\tilde{A} = (\tilde{a}_{ij})_{i,j=1}^n$ and

$$\tilde{a}_{ji} = (-1)^{i+j}\det\left([a_{lk}]_{l=1,\dots,i-1,i+1,\dots n}^{k=1,\dots,j-1,j+1,\dots n}\right).$$

We easily obtain the following formula

$$\operatorname{trace}(AA^T) = ||A||_2^2. \tag{1.1}$$

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There are well known the following formulae

$$||A|| = \max\{\sqrt{\lambda} : \det(AA^T - \lambda E) = 0\}, \tag{1.2}$$

$$||A^{-1}|| = 1/\min\{\sqrt{\lambda} : \det(AA^T - \lambda E) = 0\}, \tag{1.3}$$

where E is the identity matrix, and

$$\det(AA^T) = (\det A)^2. \tag{1.4}$$

The question arisen: what is relation between the norms $||\cdot||$ and $||\cdot||_2$ and when they coincides. In the following lemma it is given the partial answer to that question.

Lemma 1.1. *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator such that $A = [a_{ij}]_{i,j=1,\dots,n}$.*
a) There hold the inequality

$$||Ax_1 \times \cdots \times Ax_{n-1}|| \le \frac{1}{\sqrt{(n-1)^{n-1}}} ||A||_2^{n-1} ||x_1 \times \cdots \times x_{n-1}||. \tag{1.5}$$

b) If A is K quasiconformal, then

$$||Ax_1 \times \cdots \times Ax_{n-1}|| \le L(K,n) ||A||_2^{n-1} ||x_1 \times \cdots \times x_{n-1}||, \tag{1.6}$$

where

$$L(K,n) = \min\left\{ \frac{K + \sqrt{K^2-1}}{\sqrt{n^{n-2}(n-1+(K+\sqrt{K^2-1})^2)}}, \frac{1}{\sqrt{(n-1)^{n-1}}} \right\}. \tag{1.7}$$

The inequalities (1.5) *and* (1.6) *are sharp.*

Observe that $\lim_{K\to 1} L(K,n) = n^{\frac{1-n}{2}}$.

Proof. a) If $x_1, \dots, x_{n_1}$ are linearly dependent vectors, then the inequality follows from the fact that

$$Ax_1 \times \cdots \times Ax_{n-1} = \tilde{A}x_1 \times \cdots \times x_{n-1} = 0.$$

Otherwise applying Gram-Schmidt algorithm we construct a sequence of vectors $f_i,$, $i = 1, \dots, n$ such that $< f_i, f_i >= 1$, $< f_i, f_j >= 0$, for $i \neq j$ and $\mathcal{L}(x_1, \dots, x_i) = \mathcal{L}(f_1, \dots, f_i)$ for $i = 1, \dots, n-1$.

Let $F = (f_{i,j})$ be $n \times n$ matrix defined such that $f_j = \sum_{i=1}^n f_{ij} e_i$. Then

$$||AF||_2 = ||A||_2. \tag{1.8}$$

Let us prove this fact. By definition

$$||AF||_2^2 = \sum_{i,j=1}^n < A^T e_i, F e_j >^2 = \sum_{i,j=1}^n < A^T e_i, f_j >^2 .$$

Let $A^T e_i = \sum_{i,j} b_{ij} f_j$. Multiplying by f_k we obtain that $< A^T e_i, f_k >= b_{ik}$. Hence

$$A^T e_i = \sum_{i,j} < A^T e_i, f_j > f_j,$$

and consequently

$$||A^T e_i||^2 = \sum_{j=1}^{n} < A^T e_i, f_j >^2 .$$

Combining we obtain that

$$||A||_2^2 = \sum_{i=1}^{n} ||A^T e_i||^2 = \sum_{i,j=1}^{n} < A^T e_i, f_j >^2 = ||AF||_2^2.$$

Let $x_i = \sum_{j=1}^n x_{ij} f_j$, $i = 1, \dots n-1$. Then

$$Ax_1 \times \cdots \times Ax_{n-1} = \sum_\sigma \varepsilon_\sigma x_{1,\sigma_1} \dots x_{n-1\sigma_{n-1}} Af_1 \times \cdots \times Af_{n-1}.$$

It follows that

$$||Ax_1 \times \cdots \times Ax_{n-1}||^2 = ||\sum_\sigma \varepsilon_\sigma x_{1,\sigma_1} \dots x_{n-1\sigma_{n-1}} Af_1 \times \dots Af_{n-1}||^2$$

$$= ||x_1 \times \cdots \times x_{n-1}||^2 \cdot ||Af_1 \times \dots Af_{n-1}||^2 \le \frac{1}{(n-1)^{n-1}} (\sum_{i=1}^{n} ||Af_i||^2)^{(n-1)/2}$$

$$= ||x_1 \times \cdots \times x_{n-1}||^2 \cdot \frac{1}{(n-1)^{n-1}}))||AF||_2^{n/2} = ||x_1 \times \cdots \times x_{n-1}||^2 \cdot \frac{1}{(n-1)^{n-1}} ||A||_2^{n/2}.$$

If $A = (a_{ij})$ such that $a_{ii} = 1$, $i = 1, \dots, n-1$ and $a_{ij} = 0$ otherwise and $x_i = e_i$, then there hold the equality of the theorem.

b) From $Ax_1 \times \cdots \times Ax_{n-1} = \tilde{A} x_1 \times \cdots \times x_{n-1}$ it follows that

$$||Ax_1 \times \cdots \times Ax_{n-1}|| \le ||\tilde{A}|| ||x_1 \times \cdots \times x_{n-1}||.$$

According to the Proposition **??**

$$\frac{||\tilde{A}||_2^2}{||\tilde{A}||^2} = \sum_{k=1}^{n} \frac{\tilde{\lambda}_k^2}{\tilde{\lambda}_1^2} = \sum_{k=1}^{n} \frac{\tilde{\lambda}_k^2}{\tilde{\lambda}_1^2} \ge 1 + \frac{n-1}{k^2(\tilde{A})} = 1 + \frac{n-1}{k^2(\tilde{A})} \ge \frac{(K + \sqrt{K^2-1})^2 + n - 1}{(K + \sqrt{K^2-1})^2}.$$

It follows that

$$||\tilde{A}||^2 \le \frac{K + \sqrt{K^2-1}}{\sqrt{n - 1 + (K + \sqrt{K^2-1})^2}} ||\tilde{A}||_2^2 \tag{1.9}$$

On the other hand

$$||\tilde{A}||_2 = \sqrt{\sum_{k=1}^{n} ||\tilde{A} e_k||^2} = \sqrt{\sum_{k=1}^{n} ||Ae_1 \times \cdots \times Ae_{k-1} \times Ae_{k+1} \times \cdots \times Ae_n||^2}.$$

After some elementary computations it follows that

$$||\tilde{A}||_2 \le \frac{n}{\sqrt{n^n}}||A||_2^{n-1}. \tag{1.10}$$

From (1.9) and (1.10) we obtain the desired inequality. If A is the unit matrix (or more general if A is any orthogonal transformation) then A is $K = 1$-quasiconformal and the equality holds. Thus the inequality is sharp. □

Let f be a function between A and B. By $N(y, f)$ we denote the cardinal number of $f^{-1}(y)$ if the last set is finite and we set $N(y, f) = +\infty$ in the other case. The function $y \to N(y, f)$ is defined on B. If f is surjective then $N(y, f) \ge 1$ for every $y \in B$. The following proposition hold.

Proposition 1.2. [9] *Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^n$ be C^1 mapping. Then the function $y \to N(y, f)$ is measurable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} N(y, f)\,\mathrm{d}y = \int_U |J(x, f)|\,\mathrm{d}x, \tag{1.11}$$

where $J(x, f)$ is the Jacobian of f.

Further, let h be a C^1 surjection from an $n-1$ dimensional rectangle K^{n-1} onto the unit sphere S^{n-1}. Let the function f be defined in the n dimensional rectangle $K^n = [0, 1] \times K^{n-1}$ by $f(r, u) = rh(u)$. Thus f is a C^1 surjection from K^n onto the unit ball B^n. It is easily to obtain the formula $J(x, f) = r^{n-1}D_h(u)$, where $x = (r, u) \in K^n$, and D_h denotes the norm of the vector product

$$D_h = \left|\left|\frac{\partial h}{\partial x_1} \times \cdots \times \frac{\partial h}{\partial x_{n-1}}\right|\right|.$$

According to Proposition 1.2 it follows that

$$\begin{aligned}\frac{1}{n}\omega_{n-1} = \mu(B^n) &= \int_{B^n} \mathrm{d}y \le \int_{B^n} N(y, f)\,\mathrm{d}y \\ &= \int_{K^n} |J(x, f)|\,\mathrm{d}x = \int_0^1 r^{n-1}\,\mathrm{d}r \int_{K^{n-1}} D_h(u)du = \frac{1}{n}\int_{K^{n-1}} D_h(u)du.\end{aligned}$$

Consequently we have

$$\int_{K^{n-1}} D_h(u)du \ge \omega_{n-1}. \tag{1.12}$$

Proposition 1.3. *Let u be a C^1 surjection between the spherical rings $A(r_1, r_2)$ and $A(s_1, s_2)$, and let $S = u/||u||$. Let P^{n-1} be a closed $n-1$ dimensional hyper-surface that separates the components of the set $A^C(r_1, r_2)$. Then*

$$\int_{P^{n-1}} ||S'||_2^{n-1}dP \ge \sqrt{(n-1)^{n-1}}\omega_{n-1}, \tag{1.13}$$

and

$$\int_{A(r_1,r_2)} ||S'||_2^{n-1}dA \ge \sqrt{(n-1)^{n-1}}(r_2 - r_1)\omega_{n-1}, \tag{1.14}$$

where ω_{n-1} *denote the measure of* S^{n-1}.

Proof. Let K^{n-1} be an $n-1$-dimensional rectangle and let $g : K^{n-1} \to P^{n-1}$ be a parametrization of P^{n-1}. Then the function $S \circ g$ is a differentiable surjection from K^{n-1} onto the unit sphere S^{n-1}. Then by (1.12) we have

$$\int_{K^{n-1}} D_{S\circ g} dK \geq \omega_{n-1}.$$

According to Lemma 1.1 we obtain

$$\begin{aligned} D_{S\circ g}(x) &= \left|\left| S'(g(x))\frac{\partial g(x)}{\partial x_1} \times \cdots \times S'(g(x))\frac{\partial g(x)}{\partial x_{n-1}} \right|\right| \\ &\leq \frac{1}{\sqrt{(n-1)^{n-1}}} ||S'(g(x))||_2^{n-1} D_g(x). \end{aligned}$$

Hence we obtain

$$\omega_{n-1} \leq \frac{n}{\sqrt{n^n}} \int_{K^{n-1}} ||S'(g(x))||_2^{n-1} D_g(x) dK(x) = \frac{n}{\sqrt{n^n}} \int_{P^{n-1}} ||S'(\zeta)||_2^{n-1} d\sigma(\zeta).$$

Thus we have proved (1.13). It follows that

$$\int_{A(r_1,r_2)} ||S'||_2^{n-1} dA = \int_{r_1}^{r_2} \left(\int_{S^{n-1}(0,t)} ||S'||_2^{n-1} \, \mathrm{d}S \right) \mathrm{d}t \geq \sqrt{(n-1)^{n-1}} (r_2 - r_1) \omega_{n-1}.$$

The proof of the theorem has been completed. □

2. The main result

Theorem 2.1. *Let there exists a solution* u *of PDE*

$$\Delta u = f, f : \overline{A(r_1, r_2)} \mapsto \mathbb{R}^3$$

that mapps annulus $A(r_1, r_2)$ *onto annulus* $A(s_1, s_2)$ *of* $\mathbb{R}^3$, *and satisfies the conditions* $||x|| \to r_i \Rightarrow ||u(x)|| \to s_i$, $i = 1, 2$. *Note that this special PDE is the Poisson equation. Then for* $f \equiv 0$ *we have*

$$\frac{s_2}{s_1} \geq 1 - \sqrt{3} + \sqrt{3}(\log \frac{r_2}{r_1} + \frac{r_1}{r_2}) \tag{2.1}$$

and for $||f|| = \max_{r_1 \leq ||x|| \leq r_2} ||f(x)||$ *we have*

$$||f|| \geq \frac{6r_2}{(r_1 - r_2)^2} \cdot \{[3(\log \frac{r_2}{r_1} + \frac{r_1 - r_2}{r_2}) + 1]s_1 - s_2\}. \tag{2.2}$$

Note that if u is a homeomorphism then it satisfies the conditions of the theorem. Note also that (2.1) is better than inequality $\frac{s_2}{s_1} \geq \log \frac{r_2}{r_1} + \frac{r_1}{r_2}$ obtain by the author in [2]. In figure 3.1 it is shown that the inequality (2.1) is almost sharp.

Proof. Let u be a solution of given partial equation. For $n > n_0 > \max\{2, 1/(r_2 - r_1)\}$ let

$$s_n = \sup(\{||y|| : y \in A(s_1, s_2) \wedge y \notin u(A(r_1 + 1/n, r_2))\} \cup \{s_1\}).$$

If $y \in A(s_1, s_2) \wedge y \notin u(A(r_1 + 1/n, r_2))$ then $||y|| \leq s_n$ hence $y \notin A(s_n, s_2)$. Consequently $A(s_n, s_2) \subset B_n = u(A(r_1 + 1/n, r_2))$. The sequence s_n is decreasing. Hence it is a convergent. Consequently only one of the following statements hold:

(A) $s_n = s_2$ for every $n > n_0$. Then there exists a sequence $x_n : r_1 < ||x_n|| < r_1 + 1/n$ such that $||y_n|| = ||u(x_n)|| \geq s_2 - 1/n$. Since $||x_n|| \to r_1$ it follows that $||u(x_n)|| \to s_1$. This is impossible.

(B) $s_1 < s_n < s_2$ for every $n > n'$. Since u is a surjection it follows that there exists a sequence $x_n : r_1 < ||x_n|| \leq r_1 + 1/n$ such that $||y_n|| = ||u(x_n)|| = s_n$. Since $||x_n|| \to r_1$ it follows that $||u(x_n)|| = s_n \to s_1$.

(C) There exist $n'' \in \mathbb{N}$ such that $s_n = s_1$ for every $n \geq n''$.

From (A), (B) and (C) we obtain $\lim_{n\to\infty} s_n = s_1$.

Let (B) hold. For every $n > n'$, let $\varepsilon_n = s_n - s_1$ such that $s_1 + 4\varepsilon_n < s_2$ and let φ_n be a C^2 real function defined on (s_1, s_2) by

$$\varphi_n(s) = \begin{cases} s_1 & \text{if} \quad s_1 < s \leq s_1 + 2\varepsilon_n \\ h_n(s) & \text{if} \quad s_1 + 2\varepsilon_n \leq s \leq s_1 + 4\varepsilon_n \\ s_2 + \frac{s_2 - s_1 - \varepsilon_n}{s_2 - s_1 - 4\varepsilon_n}(s - s_2) & \text{if} \quad s_1 + 4\varepsilon_n \leq s < s_2 \end{cases}$$

where the function $h_n(t)$ satisfies the conditions: $h_n'(t) \geq 0$, and $h_n''(t) \geq 0$. An example of such function is the function

$$h_n(s) = s_1 + \frac{s_2 - s_1 - \varepsilon_n}{s_2 - s_1 - 4\varepsilon_n} \int_{s_1+2\varepsilon_n}^{s} \left(\frac{\int_{s_1+2\varepsilon_n}^{x} (t - s_1 - 2\varepsilon_n)(s_1 + 4\varepsilon_n - t)\ \mathrm{d}t}{\int_{s_1+2\varepsilon_n}^{s_1+4\varepsilon_n} (t - s_1 - 2\varepsilon_n)(s_1 + 4\varepsilon_n - t)\ \mathrm{d}t} \right)^q \mathrm{d}x.$$

Here $q = q_n$ is chosen such that $h_n(s_1 + 4\varepsilon_n) = s_1 + \varepsilon_n$. This is possible because $\lim_{q\to+\infty} h_n(s_1 + 4\varepsilon_n) = s_1$ and

$$h_n(s_1 + 4\varepsilon_n)|_{q=1} = s_1 + \frac{s_2 - s_1 - \varepsilon_n}{s_2 - s_1 - 4\varepsilon_n} \frac{(s_1 + 4\varepsilon_n - s_1 - 2\varepsilon_n)}{2} > s_1 + \varepsilon_n.$$

It is obvious that

$$0 \leq \varphi_n'(s) \to 1 \text{ and } 0 \leq \varphi_n''(s) \to 0 \text{ as } n \to \infty \tag{2.3}$$

for every $s \in (s_1, s_2)$. Let $\rho = ||u||$ and let ρ_n be the function defined on $\{x : r_1 < ||x|| < r_2\}$ by $\rho_n(x) = \varphi_n(\rho(x))$.

If (C) holds we can simply set $\rho_n(x) = \rho(x)$ and $\varphi_n(x) = x$.

Then

$$\Delta\rho_n(x) = \varphi_n''(\rho(x))||\Lambda\rho(x)||^2 + \varphi_n'(\rho(x))\Delta\rho(x).$$

By (2.3) it follows at once that

$$\Delta\rho_n(x) \to \Delta\rho(x) \text{ as } n \to \infty$$

for every $x \in A(r_1, r_2)$. Similarly we obtain

$$\frac{\partial\rho_n}{\partial r}(x) \to \frac{\partial\rho}{\partial r}(x) \text{ as } n \to \infty$$

uniformly on $\{x : ||x|| = r\}$ for every $r \in (r_1, r_2)$. Applying Green's formula for ρ_n on $\{x : r_1 + 1/n \leq ||x|| \leq r\}$, we obtain

$$\int_{||x||=r} \frac{\partial \rho_n}{\partial r} \,\mathrm{d}S - \int_{||x||=r_1+1/n} \frac{\partial \rho_n}{\partial r} \,\mathrm{d}S = \int_{r_1+1/n \leq ||x|| \leq r} \Delta \rho_n \, dV.$$

Since the function ρ_n is constant in some neighborhood of the sphere $||x|| = r_1 + 1/n$, it follows that

$$\int_{||x||=r} \frac{\partial \rho_n}{\partial r} \,\mathrm{d}S = \int_{r_1+1/n \leq ||x|| \leq r} \Delta \rho_n \, dV.$$

Because of (**??**) and (2.3) it follows that the function $\Delta \rho_n$ is positive for every n. Hence, by applying Fatou's lemma, letting $n \to \infty$, we obtain

$$\int_{||x||=r} \frac{\partial \rho}{\partial r} \,\mathrm{d}S \geq \int_{r_1 \leq ||x|| \leq r} \Delta \rho \, dV.$$

Next, by applying (**??**) we obtain

$$\begin{aligned} \int_{||x||=r} \frac{\partial \rho}{\partial r} \,\mathrm{d}S &\geq \int_{r_1 \leq ||x|| \leq r} \Delta \rho \, dV = \int_{r_1 \leq |z| \leq r} \rho ||S'||_2^2 + \frac{1}{2} < f, S > dV \\ &\geq s_1 \int_{r_1 \leq ||x|| \leq r} ||S'||_2^2 dV - \frac{2}{3}\pi ||f|| (r^2 - r_1^2). \end{aligned}$$

According to the relation (1.14) we obtain that:

$$r^2 \frac{\partial}{\partial r} \int_{||\zeta||=1} \rho \,\mathrm{d}S(\zeta) \geq 8\pi s_1 (r - r_1) - \frac{2}{3}\pi ||f|| (r^2 - r_1^2).$$

Dividing by r^2 and integrating over $[r_1, r_2]$ by r the previous inequality, we get

$$\begin{aligned} \int_{||\zeta||=1} \rho(r_2\zeta) \,\mathrm{d}S(\zeta) - \int_{||\zeta||=1} \rho(r_1\zeta) \,\mathrm{d}S(\zeta) \geq \\ 8\pi s_1 (\ln \frac{r_2}{r_1} + \frac{r_1 - r_2}{r_2}) - \frac{2\pi}{3} ||f|| \frac{(r_1 - r_2)^2}{r_2}. \end{aligned}$$

It follows that:

$$\begin{aligned} 4\pi(s_2 - s_1) \geq&\ 8\pi s_1 (\ln \frac{r_2}{r_1} + \frac{r_1 - r_2}{r_2}) - \frac{2\pi}{3} ||f|| \frac{(r_1 - r_2)^2}{r_2} \\ (s_2 - s_1) \geq&\ 2 s_1 (\ln \frac{r_2}{r_1} + \frac{r_1 - r_2}{r_2}) - \frac{1}{6} ||f|| \frac{(r_1 - r_2)^2}{r_2} \end{aligned}$$

hence

$$\frac{s_2}{s_1} \geq 1 + 2(\ln \frac{r_2}{r_1} + \frac{r_1 - r_2}{r_2}) - \frac{1}{6 s_1} ||f|| \frac{(r_1 - r_2)^2}{r_2}$$

□

Now the relations (2.1) and (2.2) easily follow.

3. An example

The function

$$f(x)=\left(\frac{1-r^{n-1}R}{1-r^n}+\frac{r^{n-1}R-r^n}{(1-r^n)|x|^n}\right)x \tag{3.1}$$

is a harmonic diffeomorphism between annular regions $A(r,1)$ and $A(R,1)$ if and only if

$$R\le\frac{nr}{n-1+r^n}. \tag{3.2}$$

The relation (3.2) leads us to the following conjecture: **if u is a harmonic diffeomorphism between the ring domains $A(R,1)$ and $A(r,1)$, then there hold (3.2)**. Thus we generalize the conjecture of J.C.C. Nitsche for n dimensional space.

The fact that f is diffeomorphism follows from the fact that $f(x)=\frac{x}{|x|}g(|x|)$ where

$$g(\rho)=\frac{1-r^{n-1}R}{1-r^n}\rho+\frac{r^{n-1}R-r^n}{(1-r^n)\rho^{n-1}}$$

is a diffeomorphism of $[r,1]$ onto $[R,1]$. Moreover if $h=g^{-1}$ then the function $F=\frac{x}{|x|}h(|x|)$ is the inverse mapping of f. To prove that f is harmonic mapping observe that

$$f(x)=\frac{1-r^{n-1}R}{1-r^n}x+\frac{r^{n-1}R-r^n}{1-r^n}K[\mathrm{id}](x)$$

where $K[h]$ is Kelvin transform of the mapping h defined by

$$K[h](x)=|x|^{2-n}h\left(x/|x|^2\right)$$

and it is harmonic if only if h is harmonic, see [4] for details.

Remark 3.1. a) According to the example and to the theorem 2.1 we have the inequality

$$\frac{3r}{2+r^3}\le\frac{1}{1+\sqrt{3}(r-\log r-1)}$$

for every $r\in(0,1)$. The inequality can be proved directly. See also figure 3.1
b) In [8], has been observed that if there exists a harmonic diffeomorphism between two ring domains in the space, then the modulus of co-domain cannot be small enough.

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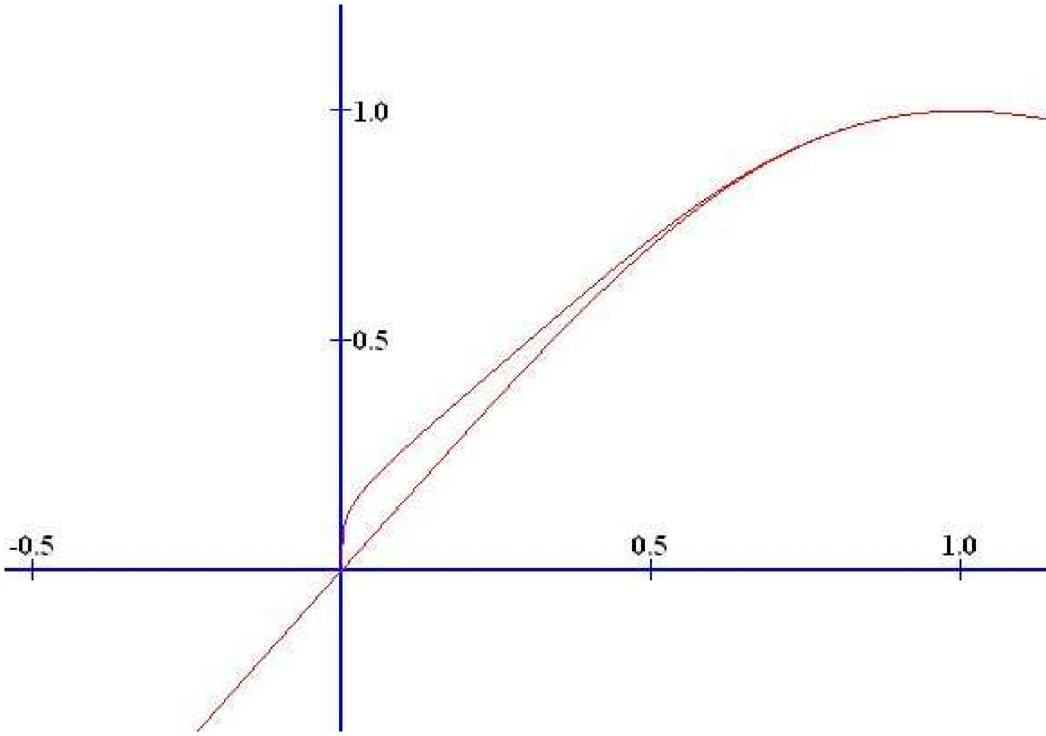
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