# A PRIORI ESTIMATE OF GRADIENT OF A SOLUTION TO CERTAIN DIFFERENTIAL INEQUALITY AND QUASICONFORMAL MAPPINGS

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ABSTRACT. We will prove a global estimate for the gradient of the solution to the *Poisson differential inequality* $|\Delta u(x)| \le a|\nabla u(x)|^2 + b$, $x \in B^n$, where $a, b < \infty$ and $u|_{S^{n-1}} \in C^{1,\alpha}(S^{n-1}, \mathbb{R}^m)$. If $m = 1$ and $a \le (n+1)/(|u|_\infty 4n\sqrt{n})$, then $|\nabla u|$ is a priori bounded. This generalizes some similar results due to E. Heinz ([13]) and Bernstein ([3]) for the plane. An application of these results yields the theorem, which is the main result of the paper: A quasiconformal mapping of the unit ball onto a domain with $C^2$ smooth boundary, satisfying the Poisson differential inequality, is Lipschitz continuous. This extends some results of the author, Mateljević and Pavlović from the complex plane to the space.

CONTENTS



## 1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

In the paper $B^n$ denotes the unit ball in $\mathbb{R}^n$, and $S^{n-1}$ denotes the unit sphere ($n > 2$). We consider the vector norm $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$ and the matrix norm $|A| = \max\{|Ax| : |x| = 1\}$. Let $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^m$ be open sets and let $u : \Omega \to \Omega'$ be a differentiable mapping. By $\nabla u$ we denote its derivative, i.e.

$$\nabla u = \begin{pmatrix} D_1 u_1 & \dots & D_n u_1 \\ \vdots & \dots & \vdots \\ D_1 u_m & \dots & D_n u_m \end{pmatrix}.$$

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If $n = m$, then the Jacobian of $u$ is defined by $J_u = \det \nabla u$. The Laplacian of a twice differentiable mapping is defined by

$$\Delta u = \sum_{i=1}^{n} D_{ii} u.$$

The solution of the equation $\Delta u = g$ (in the sense of distributions see [17]) in the unit ball, satisfying the boundary condition $u|_{S^{n-1}} = f \in L^1(S^{n-1})$, is given by

$$u(x) = \int_{S^{n-1}} P(x,\eta) f(\eta) d\sigma(\eta) - \int_{B^n} G(x,y) g(y) dy, \; |x| < 1. \tag{1.1}$$

Here

$$P(x,\eta) = \frac{1-|x|^2}{|x-\eta|^n} \tag{1.2}$$

is the Poisson kernel and $d\sigma$ is the surface $n-1$ dimensional measure of the Euclidean sphere satisfying the condition: $\int_{S^{n-1}} P(x,\eta) d\sigma(\eta) \equiv 1$. The first integral in (1.1) is called the Poisson integral and is usually denoted by $P[f](x)$. It is a harmonic mapping. The function

$$G(x,y) = c_n \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{(|1+|x|^2|y|^2 - 2\langle x,y\rangle)^{(n-2)/2}} \right), \tag{1.3}$$

where

$$c_n = \frac{1}{(n-2)\omega_{n-1}} \tag{1.4}$$

and $\omega_{n-1}$ is the measure of $S^{n-1}$, is the Green function of the unit ball. The Poisson kernel and the Green function are harmonic in $x$.

**Definition 1.1.** A homeomorfism (continuous mapping) $u : \Omega \to \Omega'$ between two open subsets $\Omega$ and $\Omega'$ of the Euclidean space $\mathbb{R}^n$ will be called a $K$ ($K \geq 1$) *quasi-conformal* (*quasi-regular*) or shortly a q.c. (q.r.) mapping if:

(i) $u$ is absolutely continuous function in almost every segment parallel to some of the coordinate axes and there exist partial derivatives which are locally $L^n$ integrable functions on $\Omega$. We will write $u \in ACL^n$.

(ii) $u$ satisfies the condition

$$\frac{|\nabla u(x)|^n}{K} \leq |J_u(x)| \leq K l(\nabla u(x))^n \quad \left( \frac{|\nabla u(x)|^n}{K} \leq J_u(x) \leq K l(\nabla u(x))^n \right) \tag{1.5}$$

for almost every $x$ in $\Omega$ where $l(u'(x)) := \inf\{|\nabla u(x)\zeta| : |\zeta| = 1\}$ and $J_u(x)$ is the Jacobian determinant of $u$ (see [33] or [37]).

We refer also to the monographs [34] and [35] for the basic theory of quasiregular mappings.

Notice that the condition $u \in ACL^n$ guarantees the existence of the first derivative of $u$ almost everywhere (see [33] ). Moreover $J_u(x) = \det(\nabla u(x)) \neq 0$ for a.e. $x \in \Omega$. For a continuous mapping $u$, the condition (i) is equivalent to the fact that $u$ belongs to the Sobolev space $W^1_{n,\text{loc}}(\Omega)$.

For a function (a mapping) $u$ defined in a domain $\Omega$ we define $|u| = |u|_\infty = \sup\{|u(x)| : x \in \Omega\}$. We say that $u \in C^{k,\alpha}(\Omega)$, $0 < \alpha \leq 1$, $k \in \mathbb{N}$, if

$$|u|_{l,\alpha} := \sum_{|\beta| \leq l} |D^\beta u| + \sum_{|\beta| = l} \sup_{x,y \in \Omega} |D^\beta u(x) - D^\beta u(y)| \cdot |x-y|^{-\alpha} < \infty.$$

It follows that for every $\alpha \in (0,1]$ and $l \in \mathbb{N}$

$$|u|_l := \sum_{|\beta|\le l} |D^\beta u| \le |u|_{l,\alpha}. \tag{1.6}$$

We first have that for $u \in C^{1,\alpha}(\Omega)$

$$|u(x) - u(y)| \le |u|_{1,\alpha}|x-y| \text{ for every } x, y \in \Omega, \tag{1.7}$$

and for real $u \in C^{1,\alpha}(\Omega)$

$$|u^2|_1 \le 2|u|_0|u|_{1,\alpha}. \tag{1.8}$$

More generally, for every real differentiable function $\tau$ and real $u \in C^{1,\alpha}(\Omega)$, we have

$$|\tau(u)|_1 \le |\tau'(u)|_0|u|_{1,\alpha}. \tag{1.9}$$

Let $\Omega$ have a $C^{k,\alpha}$ boundary $\partial\Omega$.

The norms on the space $C^{k,\alpha}(\partial\Omega)$ can be defined as follows. If $u_0 \in C^{k,\alpha}(\partial\Omega)$ then it has a $C^{k,\alpha}$ extension $u$ to the domain $\overline{\Omega}$. The norm in $C^{k,\alpha}(\partial\Omega)$ is defined by:

$$|u_0|_{k,\alpha} := \inf\{|u|_{\Omega,k,\alpha} : u|_{\partial\Omega} = u_0\}.$$

Equipped with this norm the space $C^{k,\alpha}(\partial\Omega)$ becomes a Banach space. See also [16, p. 42] for the definition of an equivalent norm in $C^{k,\alpha}(\partial\Omega)$, by using the partition of unity.

One of the starting points of this paper is the following theorem which was one of the main tools in proving some recent results of the author and Mateljevic (see [21] and [20]).

**Proposition 1.2. (Heinz-Bernstein, see** [3] **and** [13]**).** *Let $s : \overline{\mathbb{U}} \to \mathbb{R}$ ($s : \overline{\mathbb{U}} \to \overline{B^m}$) be a continuous function from the closed unit disc $\overline{\mathbb{U}}$ into the real line (closed unit ball) satisfying the conditions:*

(1) *$s$ is $C^2$ on $\mathbb{U}$,*
(2) *$s_b(\theta) = s(e^{i\theta})$ is $C^2$ with $K = \max_{\varphi\in[0,2\pi)} |\frac{\partial^2 s}{\partial\varphi^2}(e^{i\varphi})|$, and*
(3) *$|\Delta s| \le a|\nabla s|^2 + b$ on $\mathbb{U}$ for some constants $a < \infty$ ($a < 1/2$ respectively) and $b < \infty$.*

*Then the function $|\nabla s|$ is bounded on $\mathbb{U}$ by a constant $c(a, b, K)$.*

The Heinz-Bernstein theorem appeared on 1910 in the Bernstein's paper [3] and was reproved by E. Heinz on 1956 in [13]. This theorem is important in connection with the Dirichlet problem for the system

$$\Delta u = Q\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, u, x_1, x_2\right); u = (u_1(x_1,x_2), \dots, u_m(x_1,x_2)), \tag{1.10}$$

where $Q = (Q_1, \dots, Q_m)$, and $Q_j$ are quadratic polynomials in the quantities $\frac{\partial u_i}{\partial x_k}$, $i = 1, \dots, m$, $k = 1, 2$ with the coefficients depending on $u$ and $(x_1, x_2) \in \Omega$. An example of the system (1.10) is the system of differential equations that present a regular surface $S$ with fixed mean curvature $H$ with respect to isothermal parameters $(x_1, x_2)$. Also it is important in the connection with minimal surfaces and the Monge-Ampere equation.

We will consider the $n$ dimensional generalization of the system (1.10). Indeed, we will consider a bit more general situation. Assume that a twice differentiable mapping $u = u(x_1, \dots, x_n)$ satisfies the following differential inequality, which will be the main subject of the paper:

$$|\Delta u| \le a|\nabla u|^2 + b \text{ where } a, b > 0. \tag{1.11}$$

The inequality (1.11) will be called *the Poisson differential inequality.*

Recall that the harmonic mapping equations for $u = (u^1, \dots u^n) : \mathcal{N} \to \mathcal{M}$ of the Riemann manifold $\mathcal{N} = (B^n, (h_{jk}))$ into a Riemann manifold $\mathcal{M} = (\Omega, (g_{jk}))$ $(\Omega \subset \mathbb{R}^n)$ are

$$|h|^{-1/2} \sum_{\alpha,\beta=1}^{n} \partial_\alpha(|h|^{1/2} h^{\alpha\beta} \partial_\beta u^i) + \sum_{\alpha,\beta,k,\ell=1}^{n} \Gamma^i_{k\ell}(u) D_\alpha u^k D_\beta u^\ell, \ i = 1, \dots, n, \tag{1.12}$$

where ${\Gamma^i}_{k\ell}$ are Christoffel Symbols of the metric tensor $(g_{jk})$ in the target space $\mathcal{M}$:

$${\Gamma^i}_{k\ell} = \frac{1}{2} g^{im} \left( \frac{\partial g_{mk}}{\partial x^\ell} + \frac{\partial g_{m\ell}}{\partial x^k} - \frac{\partial g_{k\ell}}{\partial x^m} \right) = \frac{1}{2} g^{im} (g_{mk,\ell} + g_{m\ell,k} - g_{k\ell,m}),$$

the matrix $(g^{jk})$ ( $(h^{jk})$) is an inverse of the metric tensor $(g_{jk})$ ($(h_{jk})$), and $|h| = \det(h_{jk})$. See e.g. [18] for this definition.

**Remark 1.3.** *If Christoffel Symbols of the metric tensor $(g_{jk})$ are bounded in $\mathcal{M}$, and if the metric in $\mathcal{N}$ is conformal and bounded i.e. if $h_{jk}(x) = \rho(x)\delta_{jk}$, such that $\rho$ is bounded in $\mathcal{N}$ then $u$ satisfies* (1.11).

Note that the Poisson differential inequality is related to the problem

$$-\operatorname{div}(A(\cdot, u)\nabla u) = f(\cdot, u, \nabla u), \tag{1.13}$$

where $x \in B^n(r) := rB^n$, $u(x) \in \mathbb{R}^m$ and each $A(x, u)$, for $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ is an endomorphism on $\operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ satisfying uniformly strongly elliptic and uniformly continuous conditions; moreover $f$ satisfies the following so called *natural growth condition* (see [8])

$$|f(x, u, p)| \le a(r)|p|^2 + b \ \ (p \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^m)). \tag{1.14}$$

The problem of interior and boundary regularity of solutions to (1.13) has been treated by many authors, and among many results, it is proved that, every solution of (1.13), under some structural conditions and some conditions on the domains and initial data, has Hölder continuous extension to the boundary and is $C^{1,\alpha}$ inside. See [12] for some recent results on this topic and also [8] and [15] for earlier references.

In this paper we generalize Proposition 1.2 for the space. Namely we prove the theorems:

**Theorem A.** *Let $s : \overline{B^n} \to \mathbb{R}^m$ be a continuous function from the closed unit ball $\overline{B^n}$ into $\mathbb{R}^m$ satisfying the conditions:*

1. *$s$ is $C^2$ on $B^n$,*
2. *$s : S^{n-1} \to \mathbb{R}^m$ is $C^{1,\alpha}$ and*
3. *$|\Delta s| \le a|\nabla s|^2 + b$ on $B^n$ for some constants $b$ and $a$ satisfying the condition $2a \le |s|_\infty := \max\{|s(x)| : x \in \overline{B^n}\}$.*

*Then the function $|\nabla s|$ is bounded on $B^n$. If $a \le C_n/|s|_\infty$ where*

$$C_n = \frac{(n+1)\sqrt{\pi}\Gamma((n+1)/2)}{8n\Gamma(n/2+1)} \tag{1.15}$$

*then* $|\nabla s|$ *is bounded by a constant* $C(a, b, n, M)$*, where* $M = |\, s|_{S^{n-1}}\,|_{1,\alpha}$.

**Theorem B.** *Let* $s : \overline{B^n} \to \mathbb{R}$ *be a continuous function from the closed unit ball* $\overline{B^n}$ *into* $\mathbb{R}$ *satisfying the conditions:*

(1) *$s$ is $C^2$ on $B^n$,*
(2) *$s : S^{n-1} \to \mathbb{R}$ is $C^{1,\alpha}$ and*
(3) *$|\Delta s| \le a|\nabla s|^2 + b$ on $B^n$ for some constants $a$ and $b$.*

*Then the function* $|\nabla s|$ *is bounded on* $B^n$*.*

*If* $a \le C_n/|s|_\infty$ *then* $|\nabla s|$ *is bounded by a constant* $C(a, b, n, |\, u|_{S^{n-1}}\,|_{1,\alpha})$*.*

According Remark 1.3 it follows that Theorem A is a partial extension of [9, Theorem 4, (ii)] where it is proved a similar result, for the family of harmonic mappings $u : \mathcal{N} \to \mathcal{M}$, which map the manifold $\mathcal{N}$ into a regular ball $B_r(Q) \subset \mathcal{M}$.

Theorem A and Theorem B roughly speaking, assert that every solution to Poisson differential inequality has a Lipschitz continuous extension to the boundary, if the boundary data is $C^{1,\alpha}$.

An application of Theorem B yields the following theorem which is a generalization of analogous theorems for plane domains due to the author and Mateljevic (see [23] and [20]).

**Theorem C.** *Let* $u : B^n \to \Omega$ *be a twice differentiable quasiconformal mapping of the unit ball onto the bounded domain* $\Omega$ *with* $C^2$ *boundary, satisfying the Poisson differential inequality. Then* $\nabla u$ *is bounded and* $u$ *is Lipschitz continuous.*

**Remark 1.4.** Every $C^2$ mapping satisfies locally the Poisson differential inequality and is locally quasiconformal provided that the Jacobian is non vanishing. Thus the family of mappings satisfying the conditions of Theorem C is quite large. According to Fefferman theorem ([5]) every biholomorphism of the unit ball onto a domain with smooth boundary is smooth up to the boundary and therefore quasiconformal. This fact together with the fact that every holomorphic mapping is harmonic, it follows that Theorem C can be also considered as a partial extension of Fefferman theorem.

The paper contains this introduction and three other sections. In section 2 we prove some important lemmas. In section 3, by using Lemma 2.1 and some a priori estimates for Poisson equation proved in [10], we prove Theorem A and Theorem B, which are a priori estimates and global estimates for the the solution to Poisson differential inequality on the space, which are generalization of analogous classic Heinz-Bernstein theorem for the plane. In the final section, we previously show that, if $u : B^n \to \Omega$ is q.c. and satisfies Pisson differential inequality, then the function $\chi(x) = -d(u(x))$, where $d(u) = \operatorname{dist}(u, \partial\Omega)$, satisfies as well Pisson differential inequality in some neighborhood of the boundary. By using this fact and Theorem B we prove Theorem C. This extends some results of the author, Mateljevic and Pavlovic ([23], [26], [20], [21] and [32]) from the plane to the space. It is important to notice that, the conformal mappings and decomposition of planar harmonic mappings as the sum of an analytic and an anti-analytic function played important role in establishing some regularity boundary behaviors of q.c. harmonic mappings in the plane ([32] and [23]). This cannot be done for harmonic mappings defined in the space (see [1], [22] and [36] for some results on the topic of Euclidean and hyperbolic q.c. harmonic mappings in the space). The theorem presented here (Theorem B) made it possible to work on the problem of q.c. harmonic mappings on the space without employing the conformal and analytic functions. Notice that, the family of conformal mappings on the space coincides with Möbius transformations. Therefore this family is "smaller" than the family of conformal mappings in the plane.

## 2. Some lemmas

We will follow the approach used in [13]. The following lemma will be an essential tool in proving the main results of Section 3. It depends upon three lemmas. It is an extension of a similar result for complex plane treated in [13].

**Lemma 2.1** (The main lemma)**.** *Assumptions:*

**A1** *The mapping $u : D \to \overline{B}^m$ is $C^2$, $D \subset B^n$ satisfies for $x \in D$ the differential inequality*

$$|\Delta u| \le a|\nabla u|^2 + b \tag{2.1}$$

*where $0 < a, b < \infty$.*

**A2** *There exists a real valued function $G(u)$ of class $C^2$ for $|u| \le 1 + \epsilon$ ($\epsilon > 0$) such that*

$$|\nabla G| \le \alpha \tag{2.2}$$

*and the function $\phi(x) = G(u(x))$ satisfies the differential inequality*

$$\Delta\phi \ge \beta(|\nabla u|^2) - \gamma \tag{2.3}$$

*where $\alpha$, $\beta$ and $\gamma$ are positive constants.*

*Conclusions:*

**C** *There exists a fixed positive number $c_1' = c_1(a, b, \alpha, \beta, \gamma, n, u)$ such that for $x_0 \in D$ and $r_0 = \operatorname{dist}(x_0, \partial D)$, the following inequality holds:*

$$|\nabla u(x_0)| \le c_1' \left(1 + \frac{\max_{|x-x_0|\le r_0} |u(x) - u(x_0)|}{r_0}\right). \tag{2.4}$$

*If $a$ is small enough ($a$ satisfies the inequality $a \le C_n$ i.e. (2.32)) then $c_1'$ can be chosen independently of $u$.*

**Remark 2.2.** After writing this paper, the author learned that a similar result has been obtained in the paper [19], for the class of harmonic mappings. By using the fact that the family of bounded harmonic mappings is Hölder continuous in compacts for some Hölder exponent $\sigma < 1$ (a result obtained in [14]), Jost and Karcher proved that $c_1'$ can be chosen independently of $u$ without the previous restriction ([19, Theorem 3.1]). However, it seems that our results are new for the class of solutions to Poisson differential inequality and the author believes that $c_1'$ can be chosen independently of $u$ without the restriction $a \le C_n = \frac{(n+1)\sqrt{\pi}\Gamma((n+1)/2)}{8n\Gamma(n/2+1)}$. However to prove the main result (Theorem C) we only need an estimate that is not necessarily a priori (see Theorem B).

We will prove the lemma 2.1 by using the following three lemmas:

**Lemma 2.3.** *Let $u$ satisfy the hypotheses of the lemma 2.1 and let the ball $|y-x| \le \rho$ be contained in $D$. Then we have for $0 < \rho_1 < \rho$ the inequality*

$$\begin{aligned} &\int_{|y-x|\le\rho_1} c_n |\nabla u|^2 dy \\ &\qquad \le \frac{{\rho_1}^{n-2}\rho^{n-2}}{\rho^{n-2}-\rho_1^{n-2}}\left(\frac{\gamma\rho^2}{2n\beta}+\frac{\alpha}{\beta}\max_{|y-x|=\rho}|u(y)-u(x)|\right). \end{aligned} \tag{2.5}$$

*Proof.* By using (1.1) we obtain for $|x| < 1-\rho$

$$\begin{aligned} &\int_{S^{n-1}} (u(x+\rho\eta)-u(x))d\sigma(\eta) \\ &\qquad = \int_{|y-x|\le\rho}\left(\frac{c_n}{|y-x|^{n-2}}-\frac{c_n}{\rho^{n-2}}\right)g(y)dy. \end{aligned} \tag{2.6}$$

If we now apply the identity (2.6) to the mapping $\phi(x) = G(u(x))$, by using $|\nabla\phi| \le a$, we obtain the inequality:

$$\begin{aligned} &\int_{|y-x|\le\rho}\left(\frac{c_n}{|y-x|^{n-2}}-\frac{c_n}{\rho^{n-2}}\right)\Delta\phi dy \\ &\quad \le \alpha\int_{S^{n-1}}|u(x+\rho\eta)-u(x)|d\sigma(\eta) \le \alpha\max_{|x-y|=\rho}|u(y)-u(x)|. \end{aligned} \tag{2.7}$$

On the other hand from (2.3) we deduce

$$\begin{aligned} &\int_{|y-x|\le\rho}\Big(\frac{c_n}{|y-x|^{n-2}}-\frac{c_n}{\rho^{n-2}}\Big)\Delta\phi dy \\ &\qquad \ge \beta\int_{|y-x|\le\rho}\left(\frac{c_n}{|y-x|^{n-2}}-\frac{c_n}{\rho^{n-2}}\right)|\nabla u|^2 dy \\ &\qquad - \gamma\int_{|y-x|\le\rho}\left(\frac{c_n}{|y-x|^{n-2}}-\frac{c_n}{\rho^{n-2}}\right)dy \\ &\qquad \ge \beta\int_{|y-x|\le\rho}\left(\frac{c_n}{|y-x|^{n-2}}-\frac{c_n}{\rho^{n-2}}\right)|\nabla u|^2 dy \\ &\qquad - \frac{\gamma\rho^2}{2n}. \end{aligned} \tag{2.8}$$

Combining this inequality with (2.7) we obtain

$$\begin{aligned} &\int_{|y-x|\le\rho}\left(\frac{c_n}{|y-x|^{n-2}}-\frac{c_n}{\rho^{n-2}}\right)|\nabla u|^2 dy \\ &\qquad \le \frac{\gamma\rho^2}{2\beta n}+\frac{\alpha}{\beta}\max_{|y-x|=\rho}|u(y)-u(x)|. \end{aligned} \tag{2.9}$$

Now let $0 < \rho_1 < \rho$. From (2.9) we get

$$\begin{aligned} &\left(\frac{1}{\rho_1^{n-2}}-\frac{1}{\rho^{n-2}}\right)\int_{|y-x|\le\rho_1} c_n|\nabla u|^2 dy \\ &\qquad \le \int_{|y-x|\le\rho}\left(\frac{c_n}{|y-x|^{n-2}}-\frac{c_n}{\rho^{n-2}}\right)|\nabla u|^2 dy \\ &\qquad \le \frac{\gamma\rho^2}{2\beta n}+\frac{\alpha}{\beta}\max_{|y-x|=\rho}|u(y)-u(x)| \end{aligned}$$

and therefore

$$\int_{|y-x|\le\rho_1} c_n|\nabla u|^2 dy \le \frac{{\rho_1}^{n-2}\rho^{n-2}}{\rho^{n-2}-\rho_1^{n-2}}\left(\frac{\gamma\rho^2}{2\beta n}+\frac{\alpha}{\beta}\max_{|y-x|=\rho}|u(y)-u(x)|\right).$$

□

**Lemma 2.4.** *Let $Y : D \to B^m$ be a $C^2$ mapping of a domain $D \subset B^n$. Let $B^n(x_0,\rho)\subset D$ and let $Z\in\mathbb{R}^m$ be any constant vector ($n\ge 3$, $m\in\mathbb{N}$). Then we have the estimate:*

$$\begin{aligned}|\nabla Y(x_0)| \le{}& \frac{n}{\rho^n}\int_{|y-x_0|=\rho}|Y(y)-Z|d\sigma(y)\\ &+\frac{1}{\omega_{n-1}}\int_{|y-x_0|\le\rho}\left(\frac{1}{|y-x_0|^{n-1}}-\frac{|y-x_0|}{\rho^n}\right)|\Delta Y|dy,\end{aligned}\tag{2.10}$$

*and*

$$|\nabla Y(x_0)|\le\frac{\gamma_n}{\rho}+\frac{1}{\omega_{n-1}}\int_{|y-x_0|\le\rho}\left(\frac{1}{|y-x_0|^{n-1}}-\frac{|y-x_0|}{\rho^n}\right)|\Delta Y|dy,\tag{2.11}$$

*where $\gamma_n$ is defined in* (2.19).

*Proof.* Assume that $v\in C^2(\overline{B^n})$. From

$$v(x)=H(x)+K(x):=\int_{S^{n-1}}P(x,\eta)v(\eta)d\sigma(\eta)-\int_{B^n}G(x,y)\Delta v(y)dy\tag{2.12}$$

where $H$ is a harmonic function, it follows that

$$v'(x)h=\int_{S^{n-1}}P_x(x,\eta)h\cdot v(\eta)d\sigma(\eta)-\int_{B^n}G_x(x,y)h\cdot\Delta v(y)dy.\tag{2.13}$$

By differentiating (1.2) and (1.3) we obtain

$$P_x(x,\eta)h=\frac{-2\langle x,h\rangle}{|x-\eta|^n}-\frac{n(1-|x|^2)\langle x-\eta,h\rangle}{|x-\eta|^{n+2}}$$

and

$$G_x(x,y)h=c_n\frac{(n-2)\langle x-y,h\rangle}{|x-y|^n}-c_n\frac{(n-2)|y|^2\langle x,h\rangle-(n-2)\langle y,h\rangle}{(|1+|x|^2|y|^2-2\langle x,y\rangle)^{n/2}}.$$

Hence

$$P_x(0,\eta)h=\frac{n\langle\eta,h\rangle}{|\eta|^{n+2}}=n\langle\eta,h\rangle$$

and

$$G_x(0,y)h=-\frac{1}{\omega_{n-1}}\frac{\langle y,h\rangle}{|y|^n}+\frac{1}{\omega_{n-1}}\langle y,h\rangle.$$

Therefore

$$|P_x(0,\eta)h|\le|P_x(0,\eta)||h|=n|h|\tag{2.14}$$

and

$$|G_x(0,y)h|\le|G_x(0,y)||h|=\frac{1}{\omega_{n-1}}(|y|^{1-n}-|y|)|h|.\tag{2.15}$$

By using (2.13), (2.14) and (2.15) we obtain

$$|\nabla v(0)h| \le \int_{S^{n-1}} |P_x(0,\eta)||h||v(\eta)|d\sigma(\eta) + \int_{B^n} |G_x(0,y)||h||\Delta v(y)|dy.$$

Hence we have

$$\begin{aligned} |\nabla v(0)| &\le n \int_{S^{n-1}} |v(\eta)|d\sigma(\eta) \\ &\quad + \frac{1}{\omega_{n-1}} \int_{B^n} (|y|^{1-n} - |y|)|\Delta v(y)|dy. \end{aligned} \tag{2.16}$$

Let $v(x) = Y(x_0 + \rho x) - Z$. Then $v(0) = Y(x_0) - Z$, $\nabla v(0) = \rho \nabla Y(x_0)$ and $\Delta v(y) = \rho^2 \Delta Y(x_0 + \rho y)$. Inserting this into (2.16) we obtain

$$\begin{aligned} \rho|\nabla Y(x_0)| \le |\nabla G(0)| + |\nabla K(0)| &\le n \int_{S^{n-1}} |Y(x_0 + \rho\eta) - Z|d\sigma(\eta) \\ &\quad + \rho^2 \frac{1}{\omega_{n-1}} \int_{B^n} (|y|^{1-n} - |y|)|\Delta Y(x_0 + \rho y)|dy. \end{aligned} \tag{2.17}$$

Introducing the change of variables $\zeta = x_0 + \rho\eta$ in the first integral and $w = x_0 + \rho y$ in the second integral of (2.17) we obtain

$$\begin{aligned} |\nabla Y(x_0)| &\le \frac{n}{\rho^n} \int_{|\zeta - x_0| = \rho} |Y(\zeta) - Z|d\sigma(\zeta) \\ &\quad + \frac{1}{\omega_{n-1}} \int_{|w - x_0| \le \rho} \left( \frac{1}{|w - x_0|^{n-1}} - \frac{|w - x_0|}{\rho^n} \right) |\Delta Y| dw \end{aligned} \tag{2.18}$$

which is identical with (2.10). To get (2.11) we do as follows. We again start by (2.12). Let $v(x) = Y(x_0 + \rho x)$. Then $H$ is defined by

$$H(x) = \int_{S^{n-1}} P(x,\eta) Y(x_0 + \rho\eta) d\sigma(\eta).$$

Applying the Schwartz lemma (see [2, Theorem 6.26]) to the harmonic function $H_h : x \mapsto \langle H(x), h \rangle$, where $h$ is a unit vector in $\mathbb{R}^m$, we obtain that

$$\begin{aligned} |\nabla H(0)| &= \max_{k \in S^{n-1}, h \in S^{m-1}} |\langle \nabla H(0)k, h \rangle| = \max_{|h|=1} |\nabla H_h(0)| \\ &\le \gamma_n := \frac{2(n-1)\omega_{n-2}}{n\omega_{n-1}} = \frac{2\Gamma(1 + n/2)}{\sqrt{\pi}\Gamma((n+1)/2)} < \sqrt{n}. \end{aligned} \tag{2.19}$$

Since,

$$Y(x_0 + \rho x) = H(x) + K(x)$$

and $\nabla v(0) = \rho \nabla Y(x_0)$, by using inequality (2.19) together with the previous estimate for $|\nabla K(0)|$, it follows (2.11). □

**Lemma 2.5.** *Let $u : \overline{B^n} \to \mathbb{R}^m$ be a continuous mapping. Then there exists a positive function $\delta_u = \delta_u(\varepsilon)$, $\varepsilon \in (0, 2)$, such that if $x, y \in \overline{B^n}$, and $|x - y| < \delta_u(\varepsilon)$ then $|u(x) - u(y)| \le \varepsilon$.*

In [14, Theorem 3] is proved that the family of harmonic mappings $u : \mathcal{N} \to \mathcal{M}$, which map the monifold $\mathcal{N}$ into the regular ball $B_r(Q) \subset \mathcal{M}$ is uniformly continuous in compact subsets of $\mathcal{N}$. This implies that the function $\delta_u(\varepsilon)$ can be chosen independently on $u$. This fact can improve the conclusion of Lemma 2.1 as it is done in [19, Theorem 3.1].

*Proof of Lemma* 2.1. In order to estimate the function $|\nabla u|^2$ in the ball $|x-x_0|<r_0$ we introduce the quantity

$$M=\max_{|x-x_0|<r_0}(r_0-|x-x_0|)|\nabla u(x)|. \tag{2.20}$$

Obviously there exists a point $x_1:|x_1-x_0|<r_0$ such that

$$M=(r_0-|x_1-x_0|)|\nabla u(x_1)|. \tag{2.21}$$

Let $d=r_0-|x_1-x_0|$ and $\theta\in(0,1)$. If we apply Lemma 2.4 to the case where $Y(x)=u(x)$ and $Z=u(x_1)$, $x=x_1$ and $\rho=d\theta$, and use (2.21), we obtain

$$\begin{aligned}\frac{M}{d}&\le\min\{\frac{\gamma_n}{d\theta},\frac{n}{d^n\theta^n}\int_{|y-x_1|=d\theta}|u(y)-u(x_1)|d\sigma(y)\}\\&+\frac{1}{\omega_{n-1}}\int_{|y-x_1|\le d\theta}\left(\frac{1}{|y-x_1|^{n-1}}-\frac{|y-x_1|}{d^n\theta^n}\right)|\Delta u|dy.\end{aligned}$$

Using now (2.1) we obtain

$$\begin{aligned}\frac{M}{d}&\le\min\{\frac{\gamma_n}{d\theta},\frac{n}{d^n\theta^n}\int_{|y-x_1|=d\theta}|u(y)-u(x_1)|d\sigma(y)\}\\&+(n-2)ac_n\int_{|y-x_1|\le d\theta}\left(\frac{1}{|y-x_1|^{n-1}}-\frac{|y-x_1|}{d^n\theta^n}\right)|\nabla u|^2dy\\&+\frac{b}{\omega_{n-1}}\int_{|y-x_1|\le d\theta}\left(\frac{1}{|y-x_1|^{n-1}}-\frac{|y-x_1|}{d^n\theta^n}\right)dy.\end{aligned} \tag{2.22}$$

We shall now estimate the right hand side of (2.22). First of all, according to Lemma 2.5, we have for every $\varepsilon>0$ and $d\theta<\delta_u(\varepsilon)$ the inequality:

$$\frac{n}{d^n\theta^n}\int_{|y-x_1|=d\theta}|u(y)-u(x_1)|d\sigma(y)\le\frac{n\varepsilon}{d\theta}. \tag{2.23}$$

On the other hand

$$\frac{b}{\omega_{n-1}}\int_{|y-x_1|\le d\theta}\left(\frac{1}{|y-x_1|^{n-1}}-\frac{|y-x_1|}{d^n\theta^n}\right)dy=b\frac{n}{n+1}d\theta. \tag{2.24}$$

Next let $\lambda$ be a real number such that $0<\lambda<\theta$. Then we have the inequality

$$\begin{aligned}&\frac{a}{\omega_{n-1}}\int_{|y-x_1|\le d\theta}\left(\frac{1}{|y-x_1|^{n-1}}-\frac{|y-x_1|}{d^n\theta^n}\right)|\nabla u|^2dy\\&\le\frac{a}{\omega_{n-1}}\int_{|y-x_1|\le d\lambda}\left(\frac{1}{|y-x_1|^{n-1}}-\frac{|y-x_1|}{d^n\theta^n}\right)|\nabla u|^2dy\\&+(n-2)ad\lambda c_n\left(\frac{1}{d^n\lambda^n}-\frac{1}{d^n\theta^n}\right)\int_{|y-x_1|\le d\theta}|\nabla u|^2dy.\end{aligned} \tag{2.25}$$

In order to estimate the right hand side of this inequality we first observe that, on account of (2.21) we have for $|x-x_1|\le d\lambda$ the estimate

$$|\nabla u|^2\le\frac{M^2}{d^2(1-\lambda)^2},$$

and therefore

$$\begin{aligned}\frac{a}{\omega_{n-1}}\int_{|y-x_1|\le d\lambda}\left(\frac{1}{|y-x_1|^{n-1}}-\frac{|y-x_1|}{d^n\theta^n}\right)|\nabla u|^2dy\\ \le \frac{a}{\omega_{n-1}}\frac{M^2}{d^2(1-\lambda)^2}(\lambda d-\frac{\lambda^{n+1}d}{(n+1)\theta^n})\omega_{n-1}\\ = a\frac{M^2}{d^2(1-\lambda)^2}(\lambda d-\frac{\lambda^{n+1}d}{(n+1)\theta^n}).\end{aligned} \tag{2.26}$$

Moreover from Lemma 2.3, we conclude that

$$\begin{aligned}\frac{a}{\omega_{n-1}}\int_{|y-x_1|\le d\theta}|\nabla u|^2dy\\ \le (n-2)a\frac{d^{n-2}\theta^{n-2}}{1-\theta^{n-2}}\left(\frac{\gamma\rho^2}{2\beta n}+\frac{\alpha}{\beta}\max_{|y-x_1|=d}|u(y)-u(x_1)|\right)\\ \le (n-2)a\frac{d^{n-2}\theta^{n-2}}{1-\theta^{n-2}}\left(\frac{\gamma\rho^2}{2\beta n}+\frac{2K\alpha}{\beta}\right).\end{aligned} \tag{2.27}$$

Where $K := \max_{|x-x_0|\le r_0}|u(x)-u(x_0)|$.

Inserting now (2.26) and (2.27) in (2.25) we obtain

$$\begin{aligned}\frac{a}{\omega_{n-1}}\int_{|y-x_1|\le d\theta}\left(\frac{1}{|y-x_1|^{n-1}}-\frac{|y-x_1|}{d^n\theta^n}\right)|\nabla u|^2dy\\ \le a\frac{M^2}{d(1-\lambda)^2}(\lambda-\frac{\lambda^{n+1}}{(n+1)\theta^n})\\ +(n-2)a\lambda\left(\frac{1}{d\lambda^n}-\frac{1}{d\theta^n}\right)\frac{\theta^{n-2}}{1-\theta^{n-2}}\left(\frac{\gamma\rho^2}{2\beta n}+\frac{2K\alpha}{\beta}\right).\end{aligned} \tag{2.28}$$

Combining (2.23) (for $\theta<\delta_u(\varepsilon)/r_0$), (2.24) and (2.28) we conclude from (2.22) that the following inequality holds:

$$\begin{aligned}\frac{M}{d}\le\frac{\min\{n\varepsilon,\gamma_n\}}{d\theta}+b\frac{n}{n+1}d\theta+a\frac{M^2}{d(1-\lambda)^2}(\lambda-\frac{\lambda^{n+1}}{(n+1)\theta^n})\\ +\frac{(n-2)a}{d(1-\theta^{n-2})}\frac{\lambda}{\theta^2}\left(\left(\frac{\theta}{\lambda}\right)^n-1\right)\left(\frac{\gamma\rho^2}{2\beta n}+\frac{2K\alpha}{\beta}\right).\end{aligned} \tag{2.29}$$

Myltiplying by $d$ we get:

$$\begin{aligned}M\le\frac{\min\{n\varepsilon,\gamma_n\}}{\theta}+b\frac{n}{n+1}d^2\theta+a\frac{M^2}{(1-\lambda)^2}(\lambda-\frac{\lambda^{n+1}}{(n+1)\theta^n})\\ +\frac{(n-2)a}{(1-\theta^{n-2})}\frac{\lambda}{\theta^2}\left(\left(\frac{\theta}{\lambda}\right)^n-1\right)\left(\frac{\gamma\rho^2}{2\beta n}+\frac{2K\alpha}{\beta}\right).\end{aligned} \tag{2.30}$$

Remember that $\lambda$ and $\theta$ are arbitrary numbers satisfying $0<\lambda<\theta<1$. The inequality (2.30) can be written in the form

$$AM^2-M+B\ge 0 \tag{2.31}$$

where

$$A=a\frac{1}{(1-\lambda)^2}(\lambda-\frac{\lambda^{n+1}}{(n+1)\theta^n})$$

and

$$B = \frac{\min\{n\varepsilon, \gamma_n\}}{\theta} + b\frac{n}{n+1}d^2\theta$$
$$+ \frac{(n-2)a}{(1-\theta^{n-2})}\frac{\lambda}{\theta^2}\left(\left(\frac{\theta}{\lambda}\right)^n - 1\right)\left(\frac{\gamma\theta^2 d^2}{2\beta n} + \frac{2K\alpha}{\beta}\right).$$

Taking $\lambda = \sin\theta$ we obtain that

$$\lim_{\theta\to 0} 4AB \le \min\{\frac{4a\varepsilon n^2}{n+1}, \frac{4an\gamma_n}{n+1}\}.$$

Hence

$$4AB < 1 \text{ for } \varepsilon = \frac{n+1}{4an^2+1},$$

whenever $\theta \le \theta_0$, where $\theta_0$ is small enough. Observe that in the case

$$\frac{4an\gamma_n}{n+1} < 1, \tag{2.32}$$

$\theta_0$ can be chosen independently of $\varepsilon$ i.e. independently of $u$. The inequality (2.31) is equivalent with

$$M \le \frac{1-\sqrt{1-4AB}}{2A} = M^-(\theta) \vee M \ge \frac{1+\sqrt{1-4AB}}{2A} = M^+(\theta) \text{ for } \theta \le \theta_0. \tag{2.33}$$

From (2.33) it follows that only one of the following three cases occur:

(1) $M \le M^-(\theta)$, for $\theta \in (0, \theta_0)$;
(2) $M \ge M^+(\theta)$, for $\theta \in (0, \theta_0)$;
(3) there exist $\theta_1$, $\theta_2 \in (0, \theta_0)$ (say $\theta_1 < \theta_2$), such that $M < M^-(\theta_1)$ and $M > M^+(\theta_2) > M^-(\theta_2)$.

As $\lim_{\theta\to 0} \frac{1+\sqrt{1-4AB}}{2A} = +\infty$, the case (2) is not possible. Since $M^+$ and $M^-$ are continuous, the case (3) implies that there exists $\theta_3 \in (\theta_1, \theta_2)$ such that $M^-(\theta_3) < M < M^+(\theta_3)$. Thus the case (3) is also excluded.

The conclusion is that only the case (1) is true and henceforth

$$M \le \frac{1-\sqrt{1-4AB}}{2A} = \frac{2B}{1+\sqrt{1-4AB}} = C_2'(K, \theta_0, a, b, \alpha, \beta, \gamma, r_0, n).$$

Since $d < r_0 < 1$ it follows that $d^2 \le r_0$. Therefore

$$M \le 2B \le C_1(a, b, \alpha, \beta, \gamma, n, u)(K + r_0). \tag{2.34}$$

From $r_0|\nabla u(x_0)| \le M$ it follows the desired inequality. □

The following two lemmas, roughly speaking assert that the boundary behavior of any solution of the Poisson differential inequality is approximately the same as the boundary behavior of the set of two harmonic mappings. They are $n-$ dimensional "generalizations" of [13, Lemma 9] and [13, Lemma 9']. Since the proofs in [13] only rely on the maximum principle, the proofs of these lemmas clearly apply to $n > 2$ as well with very small modifications.

**Lemma 2.6.** *Let $u : B^n \to B^m$ be a $C^2$ mapping defined on the unit ball and satisfying the inequality*

$$|\Delta u| \le a|\nabla u|^2 + b, \tag{2.35}$$

*where* $0 < a < \frac{1}{2}$ *and* $0 < b < \infty$*. Furthermore let* $u(x)$ *be continuous for* $|x| \leq 1$*. Then we have for* $x \in B^n$ *and* $t \in S^{n-1}$ *the estimate*

$$\begin{aligned} |u(x) - u(t)| \leq \frac{1-a}{1-2a}|Y(x) - u(t)| \\ + \frac{a}{2(1-2a)}|F(x) - |u(t)|^2| + \frac{b}{2n(1-2a)}(1-|x|^2), \end{aligned} \tag{2.36}$$

*where*

$$F(x) = \int_{S^{n-1}} P(x,\eta)|u(\eta)|^2 d\sigma(\eta),\ |x| < 1 \tag{2.37}$$

*and*

$$Y(x) = \int_{S^{n-1}} P(x,\eta)u(\eta)d\sigma(\eta),\ |x| < 1. \tag{2.38}$$

**Lemma 2.7.** *Let* $\chi : \overline{B^n} \to [-1,1]$ *be a mapping of the class* $C^2(B^n) \cap C(\overline{B^n})$ *satisfying the differential inequality:*

$$|\Delta\chi| \leq a|\nabla\chi|^2 + b \tag{2.39}$$

*where* $a$ *and* $b$ *are finite constants. Then we have for* $x \in B^n$ *and* $t \in S^{n-1}$ *the estimate*

$$|\chi(x) - \chi(t)| \leq \frac{e^a}{a}\left[|h^p(x) - e^{a\chi(t)}| + |h^m(x) - e^{-a\chi(t)}| + \frac{2ab}{n}e^a(1-|x|)\right] \tag{2.40}$$

*where*

$$h^m(x) = \int_{S^{n-1}} P(x,\eta)e^{-a\chi(\eta)}d\sigma(\eta) \tag{2.41}$$

*and*

$$h^p(x) = \int_{S^{n-1}} P(x,\eta)e^{a\chi(\eta)}d\sigma(\eta). \tag{2.42}$$

## 3. A priori estimate for a solution to Poisson differential inequality

**Theorem 3.1.** *Let* $u : D \to \overline{B^m}$ *be a* $C^2$ *mapping, satisfying the differential inequality:*

$$|\Delta u| \leq a|\nabla u|^2 + b \tag{3.1}$$

*where* $0 < a < 1$ *and* $0 < b < \infty$*. Then there exists a constant* $c_2 = c_2(a,b,n,u)$ *such that for* $x_0 \in D$ *and* $r_0 = \operatorname{dist}(x_0, \partial D)$ *there holds*

$$|\nabla u(x_0)| \leq c_2\left(1 + \frac{\max_{|x-x_0|\leq r_0}|u(x) - u(x_0)|}{r_0}\right). \tag{3.2}$$

*If in addition* $a \leq C_n$ *then* $c_2$ *can be chosen independent of* $u$ *and* (3.2) *is an* **a priori estimate**.

*Proof.* Let us consider the function $G(u) = |u|^2$ and $\phi(x) = G(u(x))$. Evidently we have

$$|\nabla G(u)| = |2u| \leq 2 \text{ if } |u| \leq 1 \tag{3.3}$$

and

$$\begin{aligned} \Delta\phi &= \sum_{i=1}^m D^2G(u)(\nabla u(x)e_i, \nabla u(x)e_i) + \langle \nabla G(x), \Delta u(x)\rangle \\ &= 2|\nabla u|^2 + 2\langle u, \Delta u\rangle. \end{aligned} \tag{3.4}$$

From (3.1) we conclude

$$\Delta\phi \geq 2(1-a)|\nabla u|^2 - 2b. \tag{3.5}$$

The conditions of Lemma 2.1 are therefore satisfied by taking $\alpha = 2$, $\beta = 2(1-a)$ and $\gamma = 2b$. (3.2) follows with $c_2(a,b,n,u) = c_1(a,b,\alpha,\beta,\gamma,n,u)$. □

**Theorem 3.2.** *Let $u : \overline{B^n} \to \overline{B^m}$ be continuous in $\overline{B^n}$, $u|_{B^n} \in C^2$, $u|_{S^{n-1}} \in C^{1,\alpha}$ and satisfy the inequalities*

$$|\Delta u| \leq a|\nabla u|^2 + b, \;\; x \in B^n, \tag{3.6}$$

$$|u|_{S^{n-1}}|_{1,\alpha} \leq K, \tag{3.7}$$

*where $0 < a < 1/2$ and $0 < b, K < \infty$. Then there exists a fixed positive number $c_4(a,b,n,K,u)$ such that*

$$|\nabla u(x)| \leq c_4(a,b,n,K,u), \;\; x \in B^n. \tag{3.8}$$

*If in addition $a \leq C_n$ then $c_4(a,b,n,K,u)$ can be chosen independently of $u$ and* (3.8) *is an* **a priori estimate***.*

*Proof.* Let $x_0 = rt \in B^m$, $t \in S^{m-1}$. From Theorem 3.1 we conclude that the inequality

$$|\nabla u(x_0)| \leq c_2(a,b,n,u)\left(1 + \frac{\max_{|x-x_0|\leq 1-r}|u(x)-u(x_0)|}{1-r}\right) \tag{3.9}$$

holds.

We shall estimate the quantity

$$Q = \max_{|x-x_0|\leq 1-r}|u(x)-u(x_0)|.$$

First of all we have

$$|u(x)-u(x_0)| \leq |u(x)-u(t)| + |u(x_0)-u(t)| \;\; \text{for} \;\; |x|<1.$$

Applying now Lemma 2.6 we obtain

$$\begin{aligned} |u(x)-u(x_0)| \leq & \frac{1-a}{1-2a}(|Y(x)-u(t)| + |Y(x_0)-u(t)|) \\ & + \frac{a}{2(1-2a)}(|F(x)-|u(t)|^2| + |F(x_0)-|u(t)|^2|) \\ & + \frac{b}{2n(1-2a)}[(1-|x|^2)+(1-|x_0|^2)], \end{aligned} \tag{3.10}$$

where the harmonic functions $Y$ and $F$ are defined by (2.37) and (2.38). To continue, we use the following result due to Gilbarg and Hörmander see [10, Theorem 6.1 and Lemma 2.1],

**Proposition 3.3.** *The Dirichlet problem $\Delta u = f$ in $\Omega$, $u = u_0$ on $\partial\Omega \in C^1$ has a unique solution $u \in C^{1,\alpha}$, for every $f \in C^{0,\alpha}$, and $u_0 \in C^{1,\alpha}$, and we have*

$$|u|_{1,\alpha} \leq C(|u_0|_{1,\alpha,\partial\Omega} + |f|_{0,\alpha}) \tag{3.11}$$

*where $C$ is a constant.*

Applying (3.11) on harmonic functions $Y$ and $F$, according to (1.6), (1.7) and (1.8), we first have

$$|Y(x) - u(t)| + |Y(x_0) - u(t)| \leq 2CK(1-r) \tag{3.12}$$

and

$$|F(x) - |u(t)|^2| + |F(x_0) - |u(t)|^2| \leq 4CK(1-r). \tag{3.13}$$

Combining (3.10), (3.12) and (3.13) we obtain

$$|u(x) - u(x_0)| \leq \left[2CK\frac{1-a}{1-2a} + \frac{4CKa}{2(1-2a)} + \frac{b}{n(1-2a)}\right](1-r). \tag{3.14}$$

Thus for

$$c_3(a,b,K,n) = 2CK\frac{1-a}{1-2a} + \frac{4CKa}{2(1-2a)} + \frac{b}{n(1-2a)}$$

we have

$$Q \leq c_3(a,b,K,n)(1-r).$$

Inserting this into (3.9) we obtain

$$|\nabla u(x_0)| \leq c_2(a,b,n,u)(1 + c_3(a,b,K,n)) = c_4(a,b,K,n,u).$$

Since $x_0$ is arbitrary point of the unit ball the inequality (3.8) is established. □

Whether Theorem 3.2 holds replacing the condition $0 < a < 1/2$ by $0 < a < \infty$, is not known by the author. However adding the condition of quasiregularity we obtain the following extension of Theorem 3.2.

**Corollary 3.4.** *Assume that* $u : \overline{B^n} \to \mathbb{R}^n$ *is a* $K-$*quasiregular, twice differentiable mapping, continuous on* $\overline{B^n}$*, and* $u|_{S^{n-1}} \in C^{1,\alpha}$*. If in addition it satisfies the differential inequality*

$$|\Delta u| \leq a|\nabla u|^2 + b \text{ for some constants } a, b > 0 \tag{3.15}$$

*then*

$$|\nabla u| \leq C_8(K,a,b,u).$$

*Proof.* From (1.5) we obtain for $i = 1, \dots n$

$$\frac{1}{K} \leq \frac{l(\nabla u)^n}{J_u} \leq \frac{|\nabla u_i|_2^n}{J_u} \leq \frac{|\nabla u|^n}{J_u} \leq K \tag{3.16}$$

and hence

$$|\nabla u| \leq K^{4/n}|\nabla u_i|_2. \tag{3.17}$$

Thus for every $i = 1, \dots, n$

$$|\Delta u_i| \leq aK^{2/n}|\nabla u_i|_2^2 + b. \tag{3.18}$$

The conclusion follows according to Theorem 3.6. □

In the rest of the paper we will prove an analogous result for arbitrary $a$ and $b$. The only restriction is $u$ being a real function, i.e. $m = 1$.

**Theorem 3.5.** *Let* $B^n(x_0, r_0) \subset D \subset B^n$ *and let* $\chi : D \subset B^n \to [-1, 1]$ *be a mapping of the class* $C^2(D)$ *satisfying the differential inequality:*

$$|\Delta \chi| \leq a|\nabla \chi|^2 + b \tag{3.19}$$

*where* $a$ *and* $b$ *are finite constants. Then we have the estimate*

$$|\nabla \chi(x_0)| \leq c_5(a,b,n,\chi)\left(1 + \frac{\max_{|x-x_0|\leq r_0} |\chi(x) - \chi(x_0)|}{r_0}\right). \tag{3.20}$$

*If $a \le C_n$ then $c_5(a,b,n,\chi)$ can be chosen independently of $\chi$ and* (3.20) *is an* **a priori estimate**.

*Proof.* Let us consider a twice differentiable function $\phi(t)$, $-1 \le t \le 1$ and $\varphi(x) = \phi(\chi(x))$. The function $\varphi$ satisfies the differential equation

$$\Delta\varphi = \phi''|\nabla\chi|^2 + \phi'\Delta\chi. \tag{3.21}$$

Using (3.19) we obtain

$$\Delta\varphi \ge (\phi'' - a|\phi'|)|\nabla\chi|^2 - b|\phi'|. \tag{3.22}$$

Taking $\phi(t) = e^{2at}$ we obtain

$$\Delta\varphi \ge 2a^2e^{-2a}|\nabla\chi|^2 - 2abe^{2a}. \tag{3.23}$$

The conditions of Lemma 2.1 are therefore satisfied by taking $\alpha = 2ae^{2a}$, $\beta = 2a^2e^{-2a}$, and $\gamma = 2abe^{2a}$. Hence we conclude that (3.20) holds for $c_5(a,b,n,\chi) = c_1(a,b,\alpha,\beta,\gamma,n,\chi)$. □

**Theorem 3.6.** *Let $\chi : \overline{B^n} \to \mathbb{R}$ be continuous in $\overline{B^n}$, $\chi|_{B^n} \in C^2$, $\chi|_{S^{n-1}} \in C^{1,\alpha}$ and satisfy the inequalities*

$$|\Delta\chi| \le a|\nabla\chi|^2 + b, \;\; x \in B^n, \tag{3.24}$$

$$|\chi|_{S^{n-1}}|_{1,\alpha} \le K \tag{3.25}$$

*where $0 < a, b, K$. Then there exists a fixed positive number $c_6 = c_6(a,b,K,n,\chi)$, which do not depends on $\chi$ for $a \le C_n|/|\chi|_\infty$ such that*

$$|\nabla\chi(x)| \le c_6, \;\; x \in B^n. \tag{3.26}$$

**Remark 3.7.** The condition $|\chi|_{S^{n-1}}|_{1,\alpha} \le K$ of Theorem 3.6 is the best possible, i.e. we cannot replace it by $|\chi|_{S^{n-1}}|_1 \le K$. For example O. Martio in [28] gave an example of a harmonic diffeomorphism $w = P[f]$, of the unit disk onto itself such that $f \in C^1(S^1)$ and $\nabla w$ is unbounded. This example can be easily modified for the space. For example we can simply take $u(x_1, x_2, \dots, x_n) = P[f](x_1, x_2)$. Then $u|_{S^{n-1}} \in C^1$ but $\nabla u$ is not bounded.

*Proof.* The proof follows the same lines as the proof of Theorem 3.2. The only difference is applying Theorem 3.5 instead of Theorem 3.1 and Lemma 2.7 instead of Lemma 2.6 to the function $\chi_0 = \chi(x)/M$, where $M = \max\{|\chi(x)| : x \in \overline{B^n}\}$.

Let $x_0 = rt \in B^m$, $t \in S^{m-1}$. From Theorem 3.5 we conclude that the inequality

$$|\nabla\chi_0(x_0)| \le c_5(a,b,n,\chi)\left(1 + \frac{\max_{|x-x_0|\le r_0}|\chi_0(x) - \chi_0(x_0)|}{r_0}\right) \tag{3.27}$$

holds.

We shall estimate the quantity

$$Q = \max_{|x-x_0|\le 1-r}|\chi_0(x) - \chi_0(x_0)|.$$

First of all we have

$$|\chi_0(x) - \chi_0(x_0)| \le |\chi_0(x) - \chi_0(t)| + |\chi_0(x_0) - \chi_0(t)| \;\; \text{for} \;\; |x| < 1. \tag{3.28}$$

Applying now Lemma 2.7 we obtain

$$\begin{aligned} |\chi_0(x) - \chi_0(x_0)| \le \frac{e^a}{a}\left[|h^p(x) - e^{a\chi_0(t)}| + |h^p(x_0) - e^{a\chi_0(t)}|\right] \\ + \frac{e^a}{a}\left[|h^m(x) - e^{-a\chi_0(t)}| + |h^m(x_0) - e^{-a\chi_0(t)}|\right] \\ + \frac{2ab}{n}e^a(1 - |x| + 1 - |x_0|), \end{aligned} \tag{3.29}$$

where the harmonic functions $h^p$ and $h^m$ are defined by (2.41) and (2.42). Applying (3.11) on harmonic functions $h^p$ and $h^m$, according to (1.6), (1.7) and (1.9) for $\tau_1 = e^{at}$ and $\tau_2 = e^{-at}$, we first have

$$|h^p(x) - e^{a\chi_0(t)}| + |h^p(x_0) - e^{a\chi_0(t)}| \le 2ae^aCK(1 - r) \tag{3.30}$$

and

$$|h^m(x) - e^{-a\chi_0(t)}| + |h^m(x_0) - e^{-a\chi_0(t)}| \le 2ae^aCK(1 - r). \tag{3.31}$$

Combining (3.29)-(3.31) we obtain

$$|\chi_0(x) - \chi_0(x_0)| \le \left[4CKe^{2a} + \frac{4a^2b}{n}e^a\right](1 - r). \tag{3.32}$$

Thus for

$$c_7(a, b, K, n) = 4CKe^{2a} + \frac{4a^2b}{n}e^a$$

we have

$$Q \le C_7(a, b, K)(1 - r).$$

Inserting this into (3.27) we obtain

$$|\nabla\chi_0(x_0)| \le c_5(a, b, n, \chi)(1 + c_7(a, b, K, n)) = c_6'(a, b, K, n, \chi).$$

Since $x_0$ is an arbitrary point of the unit ball the inequality (3.26) is valid for $c_6(a, b, K, n, \chi) = c_6'(a, b, K, n, \chi) \cdot M$. □

## 4. Applications-The proof of Theorem C

4.1. **Bounded curvature and the distance function.** Let $\Omega$ be a domain in $\mathbb{R}^n$ having a non-empty boundary $\partial\Omega$. The distance function is defined by

$$d(x) = \operatorname{dist}(x, \partial\Omega). \tag{4.1}$$

The function $d$ is uniformly Lipschitz continuous and there holds the inequality

$$|d(x) - d(y)| \le |x - y|. \tag{4.2}$$

Now let $\partial\Omega \in C^2$. For $y \in \partial\Omega$, let $\nu(y)$ and $T_y$ denote respectively the unit inner normal to $\partial\Omega$ at $y$ and the tangent hyperplane to $\partial\Omega$ at $y$.

The curvature of $\partial\Omega$ at a fixed point $y_0 \in \partial\Omega$ is determined as follows. By the rotation of coordinates we can assume that $x_n$ coordinate axis lies in the direction $\nu(y_0)$. In some neighborhood $\mathcal{N}(y_0)$ of $y_0$, $\partial\Omega$ is given by $x_n = \varphi(x')$, where $x' = (x_1, \dots, x_{n-1})$, $\varphi \in C^2(T(y_0) \cap \mathcal{N}(y_0))$ and $\nabla\varphi(y_0') = 0$. The curvature of $\partial\Omega$ at $y_0$ is then described by orthogonal invariants of the Hessian matrix $D^2\varphi$ evaluated at $y_0$. The eigenvalues of $D^2\varphi(y_0')$, $\kappa_1, \dots, \kappa_{n-1}$ are called the principal curvatures of $\partial\Omega$ at $y_0$ and the corresponding eigenvectors are called the principal directions of $\partial\Omega$ at $y_0$. The mean curvature of $\partial\Omega$ at $y_0$ is given by

$$H(y_0) = \frac{1}{n-1}\sum_{i=1}^{n-1}\kappa_i = \Delta\varphi(y_0'). \tag{4.3}$$

By a further rotation of coordinates we can assume that the $x_1, \dots, x_{n-1}$ axes lie along principal directions corresponding to $\kappa_1, \dots, \kappa_{n-1}$ at $y_0$. The Hessian matrix $D^2\varphi(y_0')$ with respect to the principal coordinate system at $y_0$ described above is given by

$$D^2\varphi(y_0') = \operatorname{diag}(\kappa_1, \dots, \kappa_{n-1}).$$

**Proposition 4.1.** [11] *Let $\Omega$ be bounded domain of class $C^k$ for $k \geq 2$. Then there exists a positive constant $\mu$ depending on $\Omega$ such that $d \in C^k(\Gamma_\mu)$, where $\Gamma_\mu = \{x \in \overline{\Omega} : d(x) < \mu\}$ and for $x \in \Gamma_\mu$ there exists $y(x) \in \partial\Omega$ such that*

$$\nabla d(x) = \nu(y(x)). \tag{4.4}$$

**Proposition 4.2.** [11] *Let $\Omega$ be of class $C^k$ for $k \geq 2$. Let $x_0 \in \Gamma_\mu$, $y_0 \in \partial\Omega$ be such that $|x_0 - y_0| = d(x_0)$. Then in terms of a principal coordinate system at $y_0$, we have*

$$D^2 d(x_0) = \operatorname{diag}\Big(\frac{-\kappa_1}{1-\kappa_1 d}, \dots, \frac{-\kappa_{n-1}}{1-\kappa_{n-1}d}, 0\Big). \tag{4.5}$$

**Lemma 4.3.** *Let $\partial\Omega \in C^2$. For $x \in \Gamma_\mu$ and $y(x) \in \partial\Omega$ there holds the equation*

$$\Delta d(x) = \sum_{i=1}^{n-1}\frac{-\kappa_i(y(x))}{1-\kappa_i(y(x))d}. \tag{4.6}$$

*If for some $x_0 \in \Omega$, the mean curvature of $y_0 = y(x_0) \in \partial\Omega$, is positive: then $-d(x)$ is subharmonic in some neighborhood of $y_0$. In particular if $\Omega$ is convex then the function $\Gamma_\mu \ni x \mapsto -d(x)$ is subharmonic.*

*Proof.* The equation (4.6) follows from (4.5) and (4.9) (it is a special case of the relation (4.11) taking $u(x) = x$). If $\Omega$ is convex then for every $i$ $\kappa_i \geq 0$. Hence $\Delta(-d(x)) \geq 0$ and thus $-d(x)$ is subharmonic. □

**Lemma 4.4.** *Let $u : \Omega \to \Omega'$ be a $K$ q.r. and $\chi = -d(u(x))$. Then*

$$|\nabla\chi| \leq |\nabla u| \leq K^{2/n}|\nabla\chi| \tag{4.7}$$

*in $u^{-1}(\Gamma_\mu)$ for $\mu > 0$ such that $1/\mu > \kappa_0 = \max\{|\kappa_i(x)| : x \in \partial\Omega, i = 1, \dots, n-1\}$.*

*Proof.* Observe first that $\nabla d$ is a unit vector. From $\nabla\chi = -\nabla d \cdot \nabla u$ it follows that

$$|\nabla\chi| \leq |\nabla d||\nabla u| = |\nabla u|.$$

To continue we need the following observation. For a non-singular matrix $A$ we have

$$\begin{aligned}\inf_{|x|=1}|Ax|^2 &= \inf_{|x|=1}\langle Ax, Ax\rangle = \inf_{|x|=1}\left\langle A^tAx, x\right\rangle\\ &= \inf\{\lambda : \exists x \neq 0, A^tAx = \lambda x\}\\ &= \inf\{\lambda : \exists x \neq 0, AA^tAx = \lambda Ax\}\\ &= \inf\{\lambda : \exists y \neq 0, AA^ty = \lambda y\} = \inf_{|x|=1}|A^tx|^2.\end{aligned} \tag{4.8}$$

Next we have $\nabla\chi = -(\nabla u)^t \cdot \nabla d$ and therefore for $x \in u^{-1}(\Gamma_\mu)$, we obtain

$$|\nabla\chi| \geq \inf_{|e|=1} |(\nabla u)^t\, e| = \inf_{|e|=1} |\nabla u\, e| = l(u) \geq K^{-2/n}|\nabla u|.$$

The proof of (4.7) is completed.

□

**Lemma 4.5.** *Let $D$ and $\Omega \subset \mathbb{R}^n$ be open domains, and $\partial\Omega$ be a $C^2$ hypersurface homeomorphic to $S^{n-1}$. Let $u : D \to \Omega$ be a twice differentiable $K$ quasiregular surjective mapping satisfying the Poisson differential inequality. Let in addition $\chi(x) = -d(u(x))$. Then there exists a constant $a_1 = C(a, b, K, \Omega)$ such that*

$$|\Delta\chi(x)| \leq a_1|\nabla\chi(x)|^2 + b$$

*in $u^{-1}(\Gamma_\mu)$ for some $\mu > 0$ with $1/\mu > \kappa_0 = \max\{|\kappa_i(y)| : y \in \partial\Omega, i = 1, \dots, n-1\}$. If in addition $u$ is harmonic and $\Omega$ is convex then $\chi$ is subharmonic for $x \in u^{-1}(\Gamma_\mu)$.*

*Proof.* Let $y \in \partial\Omega$. By the considerations taken in the begin of this section we can choose an orthogonal transformation $O_y$ so that the vectors $O_y(e_i)$, $i = 1, \dots, n-1$ make the principal coordinate system in the tangent hyperplane $T_y$ of $\partial\Omega$, that determine the principal curvatures of $\partial\Omega$ and $O_y(e_n) = \nu(y)$. Let $x_0 \in B^n$. Choose $y_0 \in \partial\Omega$ so that $d(u(x_0)) = \operatorname{dist}(u(x_0), y_0)$. Take $\tilde{\partial}\Omega := O_{y_0}\partial\Omega$. Let $\tilde{d}$ be the distance function with respect to $\tilde{\partial}\Omega$. Then $d(u) = \tilde{d}(O_{y_0}(u))$ and $\chi(x) = -\tilde{d}(O_{y_0}(u(x)))$. Thus

$$\begin{aligned}\Delta\chi(x) = &-\sum_{i=1}^{n} D^2\tilde{d}(O_{y_0}(u(x)))(O_{y_0}\nabla u(x)e_i, O_{y_0}\nabla u(x)e_i)\\ &-\langle\nabla d(u(x)), \Delta u(x)\rangle.\end{aligned} \tag{4.9}$$

Next we have

$$|\langle\nabla d(u(x)), \Delta u(x)\rangle| \leq |\Delta u| \leq a|\nabla u|^2 + b. \tag{4.10}$$

Applying (4.5)

$$\begin{aligned}&\sum_{i=1}^{n} D^2\tilde{d}(O_{y_0}(u(x_0)))(O_{y_0}(\nabla u(x_0))e_i, O_{y_0}(\nabla u(x_0))e_i)\\ &\quad= \sum_{i=1}^{n}\sum_{j,k=1}^{n} D_{j,k}\tilde{d}(O_{y_0}(u(x_0)))\, D_i(O_{y_0}u)_j(x_0)\cdot D_i(O_{y_0}u)_k(x_0)\\ &\quad= \sum_{j,k=1}^{n} D_{j,k}\tilde{d}(O_{y_0}(u(x_0)))\left\langle(O_{y_0}\nabla u(x_0))^t e_j, (O_{y_0}\nabla u(x_0))^t e_k\right\rangle\\ &\quad= \sum_{i=1}^{n-1}\frac{-\tilde{\kappa}_i}{1-\tilde{\kappa}_i\tilde{d}}|(O_{y_0}\nabla u(x_0))^t e_i|^2.\end{aligned} \tag{4.11}$$

Since the principal curvatures $\tilde{\kappa}_i = \kappa_i$ are bounded by $\kappa_0$, combining (4.9), (4.10), (4.11) and (4.7), and using the relations

$$|(O_{y_0}\nabla u(x_0))^t e_i|^2 = |(\nabla u(x_0))^t O_{y_0}^t e_i|^2 \leq |\nabla u(x_0)|^2,$$

we obtain for $x \in u^{-1}(\Gamma_\mu)$

$$|\Delta\chi| \leq K^{4/n}(a + \frac{n\kappa_0}{1-\mu\kappa_0})|\nabla\chi|^2 + b, \tag{4.12}$$

which is the desired inequality. □

A $K$ q.c. self-mapping $f$ of the unit ball $B^n$ need not be Lipschitz continuous. It is holder continuous i.e. there hold the inequality

$$|f(x) - f(y)| \leq M_1(n, K)|x - y|^{K^{1/(1-n)}}. \tag{4.13}$$

See [6] for the details. See [27] for the extension of the Mori's theorem for domains satisfying the quasihyperbolic boundary conditions as well as for quasiconvex domains.

Under some additional conditions on interior regularity we obtain that a q.c. mapping is Lipschitz continuous.

**Theorem 4.6** (The main result)**.** *Let $u : B^n \to \Omega$ be a twice differentiable quasiconformal mapping of the unit ball onto the bounded domain $\Omega$ with $C^2$ boundary satisfying the Poisson differential inequality. Then $\nabla u$ is bounded and $u$ is Lipschitz continuous.*

*Proof.* From Lemma 4.5

$$|\Delta\chi| \leq a_1|\nabla\chi|^2 + b_1 \text{ for } x \in \Gamma_\mu.$$

On the other hand, by a theorem of Martio and Nyakki ([30]) $u$ has a continuous extension to the boundary. Therefore for every $x \in S^{n-1}$, $\lim_{y\to x} \chi(y) = \chi(x) = 0$. Let $\tilde{\chi}$ be an $C^2$ extension of the function $\chi|_{x\in u^{-1}(\Gamma_\mu)}$ in $B^n$ (by Whitney theorem it exists [39]). Let $b_0 = \max\{|\Delta\tilde{\chi}(x)| : x \in B^n \setminus u^{-1}(\Gamma_{\mu/2})\}$. Then

$$|\Delta\tilde{\chi}| \leq a_1|\nabla\tilde{\chi}|^2 + b_1 + b_0.$$

Thus the conditions of Theorem 3.6 are satisfied. The conclusion is that $\nabla\tilde{\chi}$ is bounded. According to (4.7) $\nabla u$ is bounded in $u^{-1}(\Gamma_\mu)$ and hence in $B^n$ as well. The conclusion of the theorem now easily follows. □

Let $u = P[f]$. Let $S = S(r, \theta) = S(r, \varphi, \theta_1, \dots, \theta_{n-2})$, $\theta \in [0, 2\pi] \times [0, \pi] \times \cdots \times [0, \pi]$ be the spherical coordinates and let $T(\theta) = S(1, \theta)$. Let in addition $x = f(T(\theta))$ and $\mathbf{n}_x$ be the normal on $\partial\Omega$ defined by the formula $\mathbf{n}_x = x_\varphi \times x_{\theta_1} \times \cdots \times x_{\theta_{n-2}}$. Since $f(S^{n-1}) = \partial\Omega$ it follows that

$$\mathbf{n}_x = |\mathbf{n}_x|\nu_x = D_x \cdot \nu_x \tag{4.14}$$

where $\nu_x$ is the unit inner normal vector that defines the tangent hyperplane of $\partial\Omega$ at $x = f(T(\theta))$

$$TP_x^{n-1} = \{y : \langle x - y, \nu_x\rangle = 0\}.$$

Since $\Omega$ is convex it follows that

$$\langle x - y, \nu_x\rangle \geq 0 \text{ for every } x \in \partial\Omega \text{ and } y \in \Omega. \tag{4.15}$$

Let in addition $u(S(r, \theta)) = (y_1, y_1, \dots, y_n)$. Then in these terms we have the following corollary.

**Corollary 4.7.** *If $u$ is a q.c. harmonic mapping of the unit ball onto a convex domain $\Omega$ with $C^2$ boundary, then:*

$$J_u \in L^\infty(B^n), \tag{4.16}$$

$$J_u^b(t) := \lim_{r\to 1} J_u(rt) \in L^\infty(S^{n-1}), \tag{4.17}$$

*and there hold the inequality*

$$J_u^b(t) \geq \frac{\operatorname{dist}(u(0), \partial\Omega)^n}{(2K)^{n^2-n}}, \tag{4.18}$$

*where $K$ is the quasiconformality constant.*

We need the following lemma.

**Lemma 4.8.** *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator such that $A = [a_{ij}]_{i,j=1,\dots,n}$. If $A$ is $K$ quasiconformal, then there hold the following double inequality*

$$K^{1-n}|A|^{n-1}|x_1 \times \cdots \times x_{n-1}| \leq |Ax_1 \times \cdots \times Ax_{n-1}| \leq |A|^{n-1}|x_1 \times \cdots \times x_{n-1}|. \tag{4.19}$$

*Here $\times \cdots \times$ denotes the vectorial product. Both inequalities in* (4.19) *are sharp.*

The author believes that the Lemma 4.8 is well-known, and its proof is given in the forthcoming author's paper [24].

*Proof.* Since (in view of Theorem 4.6) $\nabla u = (D_i u_j)_{i,j=1}^n$ is bounded, every harmonic mapping $D_i u_j$ is bounded. Therefore there exists $v_{i,j} \in L^\infty(S^{n-1})$ such that $D_i u_j = P[v_{i,j}]$. Thus $\lim_{r\to 1} D_i u_j(rt) = v_{i,j}(t)$ for every $i, j$. The relations (4.16) and (4.17) are therefore proved. On the other hand since $y_j = u_j \circ S$, $j = 1, \dots, n$, we have for a.e. $t = S(1, \theta) \in S^{n-1}$ the relations:

$$\lim_{r\to 1} y_{i\varphi}(r, \theta) = x_{i\varphi}(\theta),\ i \in \{1, \dots, n\}, \tag{4.20}$$

$$\lim_{r\to 1} y_{i\theta_j}(r, \theta) = x_{i\theta_j}(\theta),\ i \in \{1, \dots, n\}, j \in \{1, \dots, n-2\}, \tag{4.21}$$

and

$$\lim_{r\to 1} y_{ir}(r, \theta) = \lim_{r\to 1} \frac{x_i(\theta) - y_i(r, \theta)}{1 - r},\ i \in \{1, \dots, n\}. \tag{4.22}$$

From (4.20), (4.21), (4.22) and (1.1) we obtain for a.e. $t = S(1, \theta) \in S^{n-1}$:

$$\begin{aligned}
\lim_{r\to 1} J_{u\circ S}(r, \theta) &= \lim_{r\to 1} \begin{vmatrix} \frac{x_1 - y_1}{1-r} & \frac{x_2 - y_2}{1-r} & \dots & \frac{x_n - y_n}{1-r} \\ x_{1\varphi} & x_{2\varphi} & \dots & x_{n\varphi} \\ x_{1\theta_1} & x_{2\theta_1} & \dots & x_{n\theta_1} \\ \dots & \dots & \dots & \dots \\ x_{1\theta_{n-2}} & x_{2\theta_{n-2}} & \dots & x_{n\theta_{n-2}} \end{vmatrix} \\
&= \lim_{r\to 1} \int_{S^{n-1}} \frac{1+r}{|\eta - x|^n} \begin{vmatrix} x_1 - f_1(\eta) & \dots & x_n - f_n(\eta) \\ x_{1\varphi} & \dots & x_{n\varphi} \\ x_{1\theta_1} & \dots & x_{n\theta_1} \\ \dots & \dots & \dots \\ x_{1\theta_{n-2}} & \dots & x_{n\theta_{n-2}} \end{vmatrix} d\sigma(\eta) \\
&= \lim_{r\to 1} \int_{S^{n-1}} \frac{1+r}{|\eta - S(r, \theta)|^n} \langle f(T(\theta)) - f(\eta), \mathbf{n}_{f\circ T}(T(\theta)) \rangle \, d\sigma(\eta).
\end{aligned}$$

Using (4.15) and the inequality

$$\lim_{r\to 1} \frac{1+r}{|\eta - S(r, \theta)|^n} \geq \frac{1}{2^{n-1}}$$

we obtain

$$\begin{aligned}\lim_{r\to 1} J_{u\circ S}(r,\theta) &\geq \frac{D_x(\theta)}{2^{n-1}}\int_{S^{n-1}}\langle f(T(\theta))-f(\eta),\nu_x\rangle\, d\sigma(\eta)\\ &= \frac{D_x(\theta)}{2^{n-1}}\left(\langle f(T(\theta)),\nu_x\rangle - \langle u(0),\nu_x\rangle\right)\\ &= \frac{D_x(\theta)}{2^{n-1}}\langle f(T(\theta))-u(0),\nu_x\rangle\\ &= \frac{D_x(\theta)}{2^{n-1}}\mathrm{dist}\left(TP^{n-1}_{f(S(1,\theta))},u(0)\right)\\ &\geq \frac{D_x(\theta)}{2^{n-1}}\mathrm{dist}(u(0),\partial\Omega).\end{aligned}$$

Thus for a.e. $t=S(1,\theta)\in S^{n-1}$, we have

$$J^b_u(S(1,\theta))=\frac{J_{u\circ S}(\theta)}{D_T(\theta)}\geq\frac{D_x(\theta)}{D_T(\theta)}\frac{\mathrm{dist}(u(0),\partial\Omega)}{2^{n-1}}. \tag{4.23}$$

From the left side of (4.19), using the inequality

$$|\nabla u|^n\geq J_u(t)$$

we obtain

$$\frac{D_x(\theta)}{D_T(\theta)}\geq K^{1-n}|\nabla u(t)|^{n-1}\geq K^{1-n}J_u(t)^{(n-1)/n}.$$

Combining the last inequality and (4.23) we obtain (4.18). □

4.2. **An open problem.** It remains an open problem whether every q.c. harmonic mapping of the unit ball onto a domain with $C^2$ boundary is bi-Lipschitz continuous. This question has affirmative answer for the plane case (see [25]).

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