ON QUASICONFORMAL HARMONIC SURFACES WITH RECTIFIABLE BOUNDARY

DAVID KALAJ AND MIODRAG MATELJEVIĆ

Abstract. It is proved that any quasiconfomal harmonic mapping of the unit disk onto a surface with rectifiable boundary has absolutely continuous extension to the boundary. This extends the classical case for conformal mappings and minimal surfaces treated by Kellogg, Nitsche, Tsuji etc.

1. Introduction and notation

A mapping $u = (u_1, \dots, u_n) : D \to \mathbf{R}^n$ is called *harmonic* in a region $D \subset \mathbf{C}$ if for $k = 1, \dots, n$, u_k is real-valued harmonic functions in D; that is u_k is twice differentiable and satisfies the PDE

$$\Delta u_k := u_{k\,xx} + u_{k\,yy} = 0.$$

Let

$$P(r, x) = \frac{1 - r^2}{2\pi(1 - 2r\cos x + r^2)}$$

denote the Poisson kernel. Then every bounded harmonic mapping $u : \mathbf{U} \to \mathbf{R}^n$, $n \geq 1$, defined on the unit disc $\mathbf{U} := \{z : |z| < 1\}$ has the following representation

$$u(z) = P[f](z) = \int_0^{2\pi} P(r, x - \varphi) f(e^{ix}) dx, \tag{1.1}$$

where $z = re^{i\varphi}$ and f is a bounded integrable function defined on the unit circle S^1.

The Hardy space H^p (h_p) for $0 < p < \infty$ is the class of holomorphic functions $f : \mathbf{U} \to \mathbf{C}$ (harmonic mappings $u : \mathbf{U} \to \mathbf{R}^n$) on the open unit disk satisfying

$$\sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{\mathrm{i}\theta}) \right|^p \, \mathrm{d}\theta \right)^{\frac{1}{p}} < \infty.$$

$$\Big(\sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left\| u(re^{\mathrm{i}\theta}) \right\|^p \, \mathrm{d}\theta \right)^{\frac{1}{p}} < \infty. \Big)$$

A smooth mapping $u : \mathbf{U} \mapsto M^2 \subset \mathbf{R}^n$ is called K q.c. ($K \geq 1$) if

$$\|u_x\|^2 + \|u_y\|^2 \leq (K + \frac{1}{K})(\|u_x\|^2 \cdot \|u_y\|^2 - \langle u_x, u_y \rangle^2)^{1/2}, \quad z = x + iy \in \mathbf{U}. \tag{1.2}$$

If $K = 1$ then (1.2) is equivalent to the system of the equations

$$\|u_x\|^2 = \|u_y\|^2 \quad \text{and} \quad \langle u_x, u_y \rangle = 0, \tag{1.3}$$

2000 *Mathematics Subject Classification.* Primary 30C65; Secondary 31B05.
Key words and phrases. Quasiconformal maps, PDE, Lipschitz condition.

which represent isometrical (conformal) coordinates of the surface M^2. If u is harmonic and satisfies the system (1.3) then M^2 is a minimal surface. We will consider harmonic quasiconformal mappings (surfaces) and investigate their character at the boundary. These facts are well known, if w is a conformal mapping of the unit disk onto a domain with Jordan rectifiable boundary, then w as well as its inverse, has a absolutely continuous extension to the boundary and its boundary function maps the null sets onto null sets (a result of F. and M. Riesz, [13]). This result has been extended to the minimal surface by Tsuji ([15]). Concerning quasiconformal mapping in the plane Ahlfors and Beurling showed that the boundary function of a quasiconformal mapping of the unit disk onto itself need not be absolutely continuous [1]. The similar answer has been given by Heinonen for quasiconformal mappings in the space ([6] and [6]). Since quasiconformal harmonic mappings are generalizations of conformal mappings and quasiconformal harmonic surfaces are generalization of minimal surfaces, it was intrigue to establish this problem for these class of mapping. For the first class (quasiconformal harmonic mappings between plane domains) the answer is positive and this is an result of M. Mateljevic, M. Pavlović, D. Kalaj ([10]). See also [11] for the reproduction of this result. In this paper we show that, harmonic quasiconformal surface $u : \mathbf{U} = \operatorname{int} S^1 \to M^2 = \operatorname{int}\gamma \subset \mathbf{R}^n$ is absolutely continuous on the boundary, as well as its inverse function (Theorem 2.1 and Remark 2.7). Next we show that, the null sets in S^1 corresponds to the null sets in γ and the null sets in γ corresponds to the null sets in S^1 (Theorem 2.6).

Lemma 1.1. *If u is a quasiconformal mapping of the unit disk onto a surface $M^2 \subset \mathbf{R}^n$, then for $k = \frac{K^2-1}{K^2+1}$ we have*

$$\frac{|\langle u_x, u_y\rangle|}{\|u_x\|\|u_y\|} \le k < 1, \tag{1.4}$$

and

$$\frac{1}{K} \le \frac{\|u_x\|}{\|u_y\|} \le K. \tag{1.5}$$

Proof. From (1.2), denoting

$$\mu = \frac{|\langle u_x, u_y\rangle|^2}{\|u_x\|^2\|u_y\|^2}$$

and

$$\lambda = \frac{\|u_x\|}{\|u_y\|}$$

we obtain

$$\frac{1}{\lambda^2} + \lambda^2 \le (\frac{1}{K^2} + K^2)(1-\mu) - 2\mu.$$

Thus

$$\mu \le \frac{K^2 + \frac{1}{K^2} - \frac{1}{\lambda^2} - \lambda^2}{K^2 + \frac{1}{K^2} + 2} \le \frac{K^2 + \frac{1}{K^2} - 2}{K^2 + \frac{1}{K^2} + 2} = \frac{(K-1/K)^2}{(K+1/K)^2}.$$

Therefore

$$\frac{|\langle u_x, u_y\rangle|}{\|u_x\|\|u_y\|} \le \frac{K^2-1}{K^2+1}.$$

To prove (1.5) use the inequality (1.2) again. We first have

$$\|u_x\|^2 + |u_y|^2 \leq (K + \frac{1}{K})\|u_x\|\|u_y\|.$$

Dividing by $\|u_y\|^2$ we have

$$\lambda^2 - (K + \frac{1}{K})\lambda + 1 \leq 0.$$

Since $\lambda_1 = K$ and $\lambda_2 = \frac{1}{K}$ are the solutions of the equation

$$\lambda^2 - (K + \frac{1}{K})\lambda + 1 = 0,$$

it follows that

$$\frac{1}{K} \leq \frac{\|u_x\|}{\|u_y\|} \leq K$$

as desired.

□

Assume that $u : \mathbf{U} \to \mathbf{R}^n$ is a harmonic mapping defined in the unit disk $\mathbf{U}$. Consider auxiliary family of mappings $\omega_r(z) = u(rz)$. Then ω_r is harmonic and there holds $\omega_r(z) = P[g_r](z)$ where $g_r(e^{i\varphi}) = u(re^{i\varphi})$. Let $l_r = \int_0^{2\pi} ||u(re^{i\theta})||d\theta$ and choose $\rho, r < 1$.

Take $h \in R^n : ||h|| = 1$. Then

$$\begin{aligned} |\left\langle \int_0^{2\pi} P(\rho, \theta - \varphi)u(re^{i\theta})d\theta, h \right\rangle| &= |\int_0^{2\pi} P(\rho, \theta - \varphi)\left\langle u(re^{i\theta}), h\right\rangle d\theta| \\ &\leq \int_0^{2\pi} P(\rho, \theta - \varphi)|\left\langle u(re^{i\theta}), h\right\rangle| d\theta \\ &\leq \int_0^{2\pi} P(\rho, \theta - \varphi)||u(re^{i\theta})||d\theta. \end{aligned}$$

Therefore

$$\|\int_0^{2\pi} P(\rho, \theta - \varphi)u(re^{i\theta})d\theta\| \leq \int_0^{2\pi} P(\rho, \theta - \varphi)\|u(re^{i\theta})\|d\theta. \tag{1.6}$$

Using (1.6) we obtain

$$\begin{aligned} \int_0^{2\pi} ||u(r\rho e^{i\varphi})||d\varphi &= \int_0^{2\pi} \|\omega_r(\rho e^{i\varphi})\|d\varphi = \int_0^{2\pi} \|\int_0^{2\pi} P(\rho, \theta - \varphi)u(re^{i\theta})d\theta\|d\varphi \\ &\leq \int_0^{2\pi}\int_0^{2\pi} P(\rho, \theta - \varphi)\|u(re^{i\theta})\|d\theta d\varphi = \int_0^{2\pi} \|u(re^{i\varphi})\|d\varphi. \end{aligned} \tag{1.7}$$

Thus we proved the following lemma of Rado ([12]).

Lemma 1.2. *The mapping $r \to l_r$ considered before is increasing.*

Lemma 1.3. *Assume that u_1, u_2,...,u_m are harmonic mappings in the unit disk $\mathbf{U}$ and continuous in $\overline{\mathbf{U}}$. Then the function $f(z) = \sum_{i=1}^m \|u_i\|$ satisfies the maximum principle*

$$f(z) \leq \max_{|z|=1} f(z).$$

Proof. Let us show that f is subharmonic. To do so it is enough to prove that

$$\Delta f \geq 0.$$

Let S_i be defined by

$$u_i = ||u_i||S_i.$$

Since

$$\Delta||u_i|| = ||u_i||||\nabla S_i||^2,$$

it follows that

$$\Delta f = \sum_{i=1}^{n} \Delta||u_i|| \geq 0.$$

Now the conclusion of the lemma follows from the maximum principle of subharmonic functions. □

2. The main results

Theorem 2.1. *If $u(z) = P[F](z)$ is a quasiconformal harmonic mapping of the unit disk $\mathbf{U}$ onto a surface $M^2 \subset \mathbf{R}^n$ bounded by a rectifiable Jordan conture γ, then F is absolutely continuous function.*

We need the following propositions.

Proposition 2.2. [14] *For an analytic function f in the unit disk $\mathbf{U}$ to be continuous in $\overline{\mathbf{U}}$ and absolutely continuous in S^1 it is necessarily and sufficient that $f' \in H^1$. If $f' \in H^1$, then for a.e. $\theta \in [0, 2\pi)$ we have*

$$\frac{df(e^{i\theta})}{d\theta} = ie^{i\theta} f'(e^{i\theta}),$$

where

$$f'(e^{i\theta}) := \lim_{r \to 1} f'(re^{i\theta})$$

and $\frac{df(e^{i\theta})}{d\theta}$ is the derivative of the function $\theta \to f(e^{i\theta})$.

Let

$$P(r, x) = \frac{1}{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos x}$$

denote the Poisson kernel.

Proposition 2.3. [5] *For an analytic function f in the unit disk $\mathbf{U}$ to have the representation in $\mathbf{U}$ be means of Poisson integral*

$$f(re^{i\varphi}) = \int_0^{2\pi} g(e^{i\theta}) P(r, \varphi - \theta) d\theta,$$

where $g \in L^1(S^1)$ it is necessarily and sufficient that $f \in H^1(\mathbf{U})$.

Proof. Consider the function

$$l_r = \int_0^{2\pi} \left| \frac{\partial}{\partial \varphi} u(re^{i\varphi}) \right| d\varphi, \ 0 \leq r < 1.$$

Then $r \to l_r$ is increasing and is equal to the length of the smooth curve $u(S(r))$, where $S(r) = rS^1$. On the other hand the length of the curve $u(S(r))$ is equal to the limit of the following sequence when $n \to \infty$

$$s_r^n(z) = \left| u(z) - u(ze^{2\pi i/n}) \right| + \left| u(ze^{2\pi i/n}) - u(ze^{4\pi i/n}) \right| + \cdots + \left| u(ze^{2(n-1)\pi i/n}) - u(z) \right|,$$

$z : |z| = r$. By using Lemma 1.3, because the mapping u is continuous up to the boundary, we obtain

$$s_r^n(z) \le \max_{\varphi\in[0,2\pi]} \big[|u(e^{i\varphi}) - u(e^{i\varphi}e^{2\pi i/n})| + |u(e^{i\varphi}e^{2\pi i/n}) - u(e^{i\varphi}e^{4\pi i/n})| + \cdots + |u(e^{i\varphi}e^{2(n-1)\pi i/n}) - u(e^{i\varphi})|\big].$$

Letting $n \to \infty$ (because $u(S^1)$ is a rectifiable curve) we infer that $l_r < l(u(S^1)) < \infty$.

Next we have

$$u(z) = (\mathrm{Re}(a_1(z)), \mathrm{Re}(a_2(z)), \dots, \mathrm{Re}(a_n(z))),$$

where a_i are analytic functions.

It follows that

$$u_x = (\mathrm{Re}(a_1'(z)), \mathrm{Re}(a_2'(z)), \dots, \mathrm{Re}(a_n'(z))),$$

and

$$u_y = -(\mathrm{Im}(a_1'(z)), \mathrm{Im}(a_2'(z)), \dots, \mathrm{Im}(a_n'(z))).$$

On the other hand

$$\frac{\partial u}{\partial\varphi} = r u_y \cos\varphi - r u_x \sin\varphi.$$

Therefore

$$\|\frac{\partial u}{\partial\varphi}\|^2 = r^2\|u_x\|^2\sin^2\varphi + r^2\|u_x\|^2\cos^2\varphi - 2r^2\cos\varphi\sin\varphi\,\langle u_x, u_y\rangle\,.$$

By using (1.5) we obtain

$$\begin{aligned}\|\frac{\partial u}{\partial\varphi}\|^2 &\ge r^2\|u_y\|^2\sin^2\varphi + r^2\|u_x\|^2\cos^2\varphi - 2r^2\cos\varphi\sin\varphi k\|u_x\|\|u_y\|\\ &= (1-k)r^2(\|u_y\|^2\sin^2\varphi + \|u_x\|^2\cos^2\varphi) + kr^2(\cos\varphi\|u_x\| - \sin\varphi\|u_y\|)^2\\ &\ge (1-k)r^2(\|u_y\|^2\sin^2\varphi + \|u_x\|^2\cos^2\varphi)\\ &\ge r^2\frac{1-k}{2K}(\|u_x\|^2 + \|u_y\|^2) = r^2\frac{K}{1+K^2}(\|u_x\|^2 + \|u_y\|^2)\end{aligned}$$

Thus

$$\|\frac{\partial u}{\partial\varphi}\|^2 \ge r^2\frac{K}{1+K^2}(\|u_x\|^2 + \|u_y\|^2). \tag{2.1}$$

It follows that

$$|za_i'(z)| \le r(\|u_x\|^2 + \|u_y\|^2)^{1/2} \le \sqrt{\frac{1+K^2}{K}}\|\frac{\partial u}{\partial\varphi}\|.$$

Since

$$\int_0^{2\pi}\|\frac{\partial u}{\partial\varphi}\|d\varphi \le l(\gamma) < \infty$$

we infer that

$$\|\frac{\partial u}{\partial\varphi}\| \in h_1(\mathbf{U}).$$

Therefore for all $i = 1, \dots, n$ we have

$$za_i'(z) \in H^1(\mathbf{U})$$

and consequently

$$a_i'(z) \in H^1(\mathbf{U}).$$

By using Proposition 2.2 and Proposition 2.3 it follows that for every $i = 1, \dots, n$ there exists an absolutely continuous function g_i such that

$$a_i(z) = P[g_i(e^{i\theta})](z),\ i = 1, \dots, n.$$

Henceforth

$$u = P[F](z)$$

where

$$F(e^{i\theta}) = (\operatorname{Re} g_1(e^{i\theta}), \dots, \operatorname{Re} g_n(e^{i\theta}))$$

in an absolutely continuous function.

□

Corollary 2.4. *It follows from the previous considerations, together with the Fatou's lemma the relation*

$$\lim_{r\to 1} l_r = \int_0^{2\pi} \|\frac{\partial F}{\partial \varphi}\| d\varphi = \mu(\gamma).$$

Remark 2.5. Since quasiconformal harmonic surface has the representation

$$u(z) = (\operatorname{Re}(a_1(z)), \operatorname{Re}(a_2(z)), \dots, \operatorname{Re}(a_n(z))),$$

it follows from that our quasiconformal surface defines a minimal surface

$$w(z) = (a_1(z), a_2(z), \dots, a_n(z)) : \mathbf{U} \to \mathbf{C}^n.$$

Namely

$$||w_x||^2 = \sum_{k=1}^n |a_k'(z)|^2 = \sum_{k=1}^n |ia_k'(z)|^2 = ||w_y||^2$$

and

$$\langle w_x, w_y \rangle = \sum_{k=1}^n \operatorname{Re}(ia_k'(z)\overline{a_k'(z)}) = 0.$$

According to (2.1) minimal surface w is also a surface with rectifiable boundary, and thus it has absolutely continuous extension to the boundary. It is well-known the following isoperimetric inequality for minimal surfaces

$$A(w) \le \frac{1}{4\pi} L^2(w),$$

where

$$A(w) = \frac{1}{2} \int_{\mathbf{U}} |w_x|^2 + |w_y|^2 dxdy$$

and

$$L(w) = \int_{S^1} |dw|.$$

Since

$$A(u) = \int_{\mathbf{U}} \sqrt{|u_x|^2 |u_y|^2 - \langle u_x, u_y \rangle} du$$

it follows that

$$\begin{aligned}\frac{K}{2K^2+2}\int_{\mathbf{U}}(|u_x|^2+|u_y|^2)dxdy \leq A(u)\\ \leq \frac{1}{2}\int_{\mathbf{U}}(|u_x|^2+|u_y|^2)dxdy\\ = \frac{1}{4}\int_{\mathbf{U}}|w_x|^2+|w_y|^2dxdy\\ = \frac{1}{2}A(w).\end{aligned}$$

$$\frac{A(u)}{L^2(u)}.$$

Theorem 2.6. *If $u = P[F]$ is quasiconformal harmonic of the unit disk $\mathbf{U}$ onto a surface M^2 with rectifiable boundary γ, then for every Lebesgue measured set $E \subset S^1$ $F(E) \subset \gamma$ is measured and*

$$\mu(E) = 0 \Leftrightarrow \mu(F(E)) = 0.$$

Proof. Assume that $\mu(E) = 0$. Prove that $\mu(F(E)) = 0$. Since F is absolutely continuous function it follows that

$$\mu(\gamma) = \int_0^{2\pi}\|\frac{\partial F}{\partial\varphi}\|d\varphi.$$

Thus

$$\mu(E) = 0 \Rightarrow \mu(F(E)) = 0.$$

On the other hand by (2.1)

$$\|\frac{\partial F}{\partial\varphi}(e^{i\varphi})\| = \lim_{z\to e^{i\varphi}}\|\frac{\partial u}{\partial\varphi}(re^{i\varphi})\| \geq \frac{K}{1+K^2}\sum_{i=1}^n|a_i'(e^{i\varphi})|^2.$$

By Luzin-Privalov uniqueness theorem

$$\sum_{i=1}^n|a_i'(e^{i\varphi})|^2 > 0, \text{ for a.e. } \varphi\in[0,2\pi]$$

if at least one of the analytic functions a_i is nonconstant.

Therefore

$$\|\frac{\partial F}{\partial\varphi}(e^{i\varphi})\| > 0, \text{for a.e. } \varphi\in[0,2\pi]. \tag{2.2}$$

Next we will prove that a null set on γ corresponds to a null set on S^1. Let E be a null set on γ which corresponds to K on S^1 and E' be a null set which contains E and is G_δ, which corresponds to K' on S^1. Then E' contains E and being the continuous image of G_δ is G_δ and hence is measurable. Hence if we deduce $\mu(E') = 0$ from $\mu(K') = 0$, then $\mu(E) = 0$ follows a fortiori, so that we assume that E is measurable. Since $\mu(K) = 0$, we can cover E by a sequence of open intervals Δs_n such that $\sum_{k=1}^\infty|\Delta s_k| < \varepsilon$ where $|\Delta s_n|$ denotes the arc length of Δs_n. Let $\Delta\theta_n$ correspondents to Δs_n on S^1, then

$$|\Delta s_n| = \int_{\Delta\theta_n}\|\frac{\partial F}{\partial\varphi}(e^{i\varphi})\|d\varphi,$$

and therefore

$$\varepsilon > \sum_{k=1}^{\infty} |\Delta s_k| = \sum_{k=1}^{\infty} \int_{\Delta\theta_n} \|\frac{\partial F}{\partial \varphi}(e^{i\varphi})\| d\varphi \geq \int_E \|\frac{\partial F}{\partial \varphi}(e^{i\varphi})\| d\varphi.$$

Since ε is arbitrary, we have

$$\int_E \|\frac{\partial F}{\partial \varphi}(e^{i\varphi})\| d\varphi = 0.$$

From (2.2) we infer $\mu(E) = 0$ as desired.

□

Remark 2.7. The previous theorem implies that $u^{-1} : M^2 \to \mathbf{U}$ is absolutely continuous on γ; that is for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{k=1}^{m} |\Delta s_k| < \delta \Rightarrow \sum_{k=1}^{m} |\Delta \theta_k| < \varepsilon.$$

References

1. Beurling, A.; Ahlfors, L. *The boundary correspondence under quasiconformal mappings.* Acta Math. 96 (1956), 125-142.
2. R. Courant: *Dirichlet's Principle.* New York: Interscience 1950.
3. N.N. Luzin, I.I. Privalov, *Sur l'unicit et la multiplicit des fonctions analytiques* Ann. Sci. Ecole Norm. Sup. (3) , 42 (1925) pp. 143-191.
4. L. Simon, *A Hölder estimate for quasiconformal maps between surfaces in Euclidean space.* Acta Math. 139 (1977), no. 1-2, 19–51.
5. G. M. Goluzin, *Geometrical function theory*, Nauka Moskva 1966.(Russian)
6. J. Heinonen, *The boundary absolute continuity of quasiconformal mappings.* Amer. J. Math. 116 (1994), no. 6, 1545–1567.
7. J. Heinonen, *The boundary absolute continuity of quasiconformal mappings.* II. Rev. Mat. Iberoamericana 12 (1996), no. 3, 697–725
8. Nitsche, Johannes C. C. *The boundary behavior of minimal surfaces. Kellogg's theorem and Branch points on the boundary.* Invent. Math. 8 1969 313–333.
9. Nitsche, Johannes C. C. On new results in the theory of minimal surfaces. Bull. Amer. Math. Soc. 71 1965 195–270.
10. D. Kalaj: *Harmonic functions and harmonic quasiconformal mappings between convex domains,* Thesis, Beograd 2002.
11. D. Partyka, K. Sakan, *On bi-Lipschitz type inequalities for quasiconformal harmonic mappings.* Ann. Acad. Sci. Fenn. Math. 32 (2007), no. 2, 579–594.
12. Radó, Tibor *On Plateau's problem.* Ann. of Math. (2) 31 (1930), no. 3, 457–469.
13. F. and M. Riesz *Über die Randwerte einer analytischen Funktion.* Quatriéme congres des mathématiciens scandinaves á Stockholm, 1916.
14. F. Riesz *Uber die Randwerte einer analytischen Funktionen,* Math. Z. 18, (1923) 87-95.
15. Tsuji, Masatsugu *On a theorem of F. and M. Riesz.* Proc. Imp. Acad. Tokyo 18, (1942) 172–175.
16. M. Pavlović: *Boundary correspondence under harmonic quasiconformal homeomorfisms of the unit disc*, Ann. Acad. Sci. Fenn., Vol 27, 2002, 365-372.

University of Montenegro, Faculty of Natural Sciences and Mathematics, Cetinjski put b.b. 81000 Podgorica, Montenegro
E-mail address: `davidk@cg.yu`

University of Belgrade, Faculty of Mathematics, Studentski trg 16, 11000 Belgrade, Serbia
E-mail address: `miodrag@matf.bg.ac.yu`