AN ISOPERIMETRIC TYPE INEQUALITY FOR HARMONIC FUNCTIONS

DAVID KALAJ AND ROMEO MEŠTROVIĆ

ABSTRACT. Let h^2 be the harmonic Hardy space and b^4 be the harmonic Bergman space of harmonic functions on the open unit disk $\mathbb{U}$. Then for any function f in h^2 we prove the following isoperimetric type inequality

$$\|f\|_{b^4} \le \sqrt[4]{\frac{3+2\sqrt{2}}{2}}\|f\|_{h^2},$$

where $\|\cdot\|_{b^4}$ and $\|\cdot\|_{h^2}$ are norms in the spaces b^4 and h^2, respectively.

1. Introduction

Throughout the paper we let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, and let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in $\mathbb{C}$. The normalized area measure on $\mathbb{U}$ will be denoted by $d\sigma$. In terms of real (rectangular and polar) coordinates, we have

$$d\sigma = \frac{1}{\pi}dxdy = \frac{1}{\pi}rdrd\theta, \quad z = x + iy = re^{i\theta}.$$

Further, $dt/2\pi$ denotes the normalized Lebesgue measure on $\mathbb{T}$.

For $1 \le p < +\infty$ let $L^p(\mathbb{U}, \sigma) = L^p$ denote the familiar Lebesgue space on $\mathbb{U}$ with respect to the measure σ. For such a p, the *Bergman space* A^p is the space of all holomorphic functions on $\mathbb{U}$ such that

$$\|f\|_p := \left(\int_{\mathbb{U}} |f(z)|^p \, d\sigma\right)^{1/p} < +\infty. \tag{1.1}$$

We denote by A_0^p the set of all functions $f \in A^p$ for which $f(0) = 0$. Recall that the *harmonic Bergman space* b^p is the space of all (complex) harmonic functions f on the disk $\mathbb{U}$ such that the integral in (1.1) is finite.

The harmonic *Hardy space* h^p is defined as the space of (complex) harmonic functions f on $\mathbb{U}$ such that

$$\|f\|_{h^p} := \sup_{0\le r<1}\left(\int_0^{2\pi} |f(re^{it})|^p \, \frac{dt}{2\pi}\right)^{1/p} < \infty. \tag{1.2}$$

If $f \in h^p$ then by [1, Theorem 6.13] the radial limit

$$f(e^{it}) = \lim_{r\to 1} f(re^{it})$$

2000 *Mathematics Subject Classification.* 35A25, 30D55.
Key words and phrases. Bergman space, Hardy space, Isoperimetric inequality.

exists for almost every e^{it} in $\mathbb{T}$ and the boundary function $f(e^{it})$ is integrable on $\mathbb{T}$. It is well known that

$$\|f\|_{h^p}^p = \lim_{r\to 1}\int_0^{2\pi} |f(re^{it})|^p \,\frac{dt}{2\pi} = \int_0^{2\pi} |f(e^{it})|^p \,\frac{dt}{2\pi}.$$

The *Hardy space* H^p consists of all holomorphic functions $f \in h^p$. For more information on Bergman and Hardy spaces, see the book [5].

The starting point of this paper is the well known isoperimetric inequality for Jordan domains and isoperimetric inequality for minimal surfaces due to Carleman [3]. Among the other results, Carleman in [3] proved that if u is a harmonic and smooth function on the closed disk $\overline{\mathbb{U}} : |z| \le 1$, then

$$\int_{\mathbb{U}} e^{2u}\, dxdy \le \frac{1}{4\pi}\left(\int_0^{2\pi} e^u\, dt\right)^2.$$

By using a similar approach as Carleman, Strebel in [7] proved that if f is in H^1 then

$$\int_{\mathbb{U}} |f(z)|^2\, dxdy \le \frac{1}{4\pi}\left(\int_{\mathbb{T}} |f(e^{it})|\, dt\right)^2 \tag{1.3}$$

with "=" instead of "≤" if and only if

$$f(z) = \frac{\alpha}{1-az},$$

where $|a| < 1$, $\alpha \in \mathbb{C}$. This inequality has been proved independently by Mateljević and Pavlović in [6].

It is useful to observe that for our purposes the inequality (1.3) may be written in terms of the A^2 and H^1 norms as

$$\|f\|_{A^2} \le \|f\|_{H^1}, \quad f \in H^1. \tag{1.4}$$

Further, Burbea [2] generalized the inequality (1.3) as

$$\frac{n-1}{\pi}\int_{\mathbb{U}} |f(z)|^{np}(1-|z|^2)^{n-2}\, dxdy \le \left(\frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^p\, dt\right)^n,$$

where $n \ge 2$ is a positive integer and $f \in H^p$ for some $0 < p < +\infty$. Recently, Hang, Wang and Yang [4] extended the above type inequality for harmonic functions defined on the unit ball of $\mathbb{R}^n$ with $n \ge 3$.

Although we are unable to establish the harmonic version of the inequality (1.3), in this paper we prove its h^2–analogue as follows.

Theorem 1.1 (The main result)**.** *Suppose f is a nonzero function in the space h^2. Then*

$$\int_{\mathbb{U}} |f(z)|^4\, dxdy < \frac{3+2\sqrt{2}}{8\pi}\left(\int_{\mathbb{T}} |f(e^{it})|^2\, dt\right)^2. \tag{1.5}$$

Remark 1.2. Note that by (1.5), it follows that $h^2 \subset b^4$. Further, in terms of the b^4 and h^2 norms of the function $f \in h^2$, the inequality (1.5) can be written in the form

$$\|f\|_{b^4} \le \sqrt[4]{\frac{3+2\sqrt{2}}{2}}\|f\|_{h^2},$$

with the estimate $\sqrt[4]{(3+2\sqrt{2})/2} \approx 1.306563$.

Remark 1.3. The question arises can we replace in (1.5) $|f(z)|^4$ and $|f(e^{it})|^2$ by $|f(z)|^{2p}$ and $|f(e^{it})|^p$, respectively, for some $p > 0$. It can be shown that for $p > 1$ there holds an isoperimetric type inequality, but for $p = 1$ the answer is negative (see Example 1.4 below). Furthermore, it remains an open question whether the inequality (1.5) is sharp. The function from Example 1.5 shows that the best constant in (1.5) is greater than or equal to $5/8\pi$.

Example 1.4. Let for $|a| < 1$

$$f_a(z) = \frac{1-|a|^2|z|^2}{|1-za|^2}.$$

Then

$$\int_{\mathbb{U}} |f_a(z)|^2 d\sigma \to \infty$$

and

$$\int_{\mathbb{T}} |f_a(e^{it})| dt = 2\pi.$$

Example 1.5. Let for $|a| < 1$

$$f_a(z) = \mathrm{Re}\,\frac{z}{1-az}.$$

Then for $a \to 1-$

$$\int_{\mathbb{U}} |f_a(z)|^4 dxdy : \left(\frac{3+2\sqrt{2}}{8\pi}\left(\int_{\mathbb{T}} |f_a(e^{it})|^2 dt\right)^2\right) \to \frac{5}{3+2\sqrt{2}} \approx 0.857864.$$

2. Preliminaries

In order to prove Theorem 1.1, we will need some auxiliary results.

Lemma 2.1. *For any complex number z there holds*

$$|z\mathrm{Re}\,(z)| = \frac{\sqrt{2}}{2}|z|\sqrt{|z|^2 + \mathrm{Re}\,(z^2)}. \tag{2.1}$$

Proof. An easy calculation shows that

$$|z^2 + |z|^2| = \sqrt{2}|z|\sqrt{|z|^2 + \mathrm{Re}\,(z^2)}.$$

Now (2.1) follows from the previous identity and the identity $|z^2 + |z|^2| = 2|z\mathrm{Re}\,(z)|$. □

Lemma 2.2. *Let h be a function in A_0^1. Then*

$$\int_{\mathbb{U}} h(z)\,d\sigma = \int_{\mathbb{U}} \bar{h}(z)\,d\sigma = \int_{\mathbb{U}} \mathrm{Re}\,(h(z))\,d\sigma = 0. \tag{2.2}$$

Proof. Since $h(0)=0$, we can write $h(z)=\sum_{n=1}^{\infty} a_n z^n$. Therefore, the first two equalities in (2.2) immediately follow from the fact that $\int_0^{2\pi} e^{int}\,dt = 0$ for each $n=\pm1\pm2,\cdots$. From this and the identity $2\mathrm{Re}\,(h(z)) = h(z)+\bar{h}(z)$ it follows that $\int_{\mathbb{U}} \mathrm{Re}(h(z))\,d\sigma = 0$. □

Lemma 2.3. *If h is a function in the space A_0^2, then*

$$\int_{\mathbb{U}} |h(z)|^2\,d\sigma = 2\int_{\mathbb{U}} |\mathrm{Re}\,(h(z))|^2\,d\sigma. \tag{2.3}$$

Proof. Since

$$\begin{aligned}
|\mathrm{Re}\,(h(z))|^2 &= \left|\frac{h(z)+\bar{h}(z)}{2}\right|^2 = \frac{1}{4}(h(z)+\bar{h(z)})(\bar{h}(z)+h(z)) \\
&= \frac{1}{4}(2|h(z)|^2 + h^2(z) + \bar{h}^2(z)) = \frac{1}{2}|h(z)|^2 + \frac{1}{2}\mathrm{Re}\,(h^2(z)),
\end{aligned}$$

by integrating this and applying the fact that by Lemma 2.2, $\int_{\mathbb{U}} \mathrm{Re}\,(h^2(z))\,d\sigma = 0$, we obtain (2.3). □

Lemma 2.4. *Let h be a function in the space A_0^2 that is not identically zero on $\mathbb{U}$. Then*

$$\int_{\mathbb{U}} |h(z)\mathrm{Re}\,(h(z))|\,d\sigma < \frac{\sqrt{2}}{2}\int_{\mathbb{U}} |h(z)|^2\,d\sigma. \tag{2.4}$$

Proof. By the identity (2.1) of Lemma 2.1, we have

$$|h(z)\mathrm{Re}\,(h(z))| = \frac{\sqrt{2}}{2}|h(z)|\sqrt{|h(z)|^2 + \mathrm{Re}\,(h^2(z))}. \tag{2.5}$$

By applying Cauchy-Schwartz inequality and the fact that by Lemma 2.2, $\int_{\mathbb{U}} \mathrm{Re}\,(h^2(z))\,d\sigma = 0$, we obtain

$$\begin{aligned}
&\Big(\int_{\mathbb{U}} |h(z)|\sqrt{|h(z)|^2 + \mathrm{Re}\,(h^2(z))}\,d\sigma\Big)^2 \\
&\quad\le \int_{\mathbb{U}} |h(z)|^2\,d\sigma \left(\int_{\mathbb{U}} (|h(z)|^2 + \mathrm{Re}\,(h^2(z)))\,d\sigma\right) \\
&\quad= \left(\int_{\mathbb{U}} |h(z)|^2\,d\sigma\right)^2 + \left(\int_{\mathbb{U}} |h(z)|^2\,d\sigma\right)\left(\int_{\mathbb{U}} \mathrm{Re}\,(h^2(z))\,d\sigma\right) \\
&\quad= \left(\int_{\mathbb{U}} |h(z)|^2\,d\sigma\right)^2.
\end{aligned}$$

The above inequality and (2.5) immediately yield the desired inequality (2.4) with $\le$ instead of $<$. In order to show the strong inequality, we first observe that the equality in the previously applied Cauchy-Schwartz inequality holds if and only if

$$|h^2(z)| = \lambda\sqrt{|h(z)|^2 + \mathrm{Re}\,(h^2(z))} \tag{2.6}$$

for almost every $z \in \mathbb{U}$ and a nonnegative constant λ. If $\lambda = 0$ then obviously, we have $h \equiv 0$ on $\mathbb{U}$. If $\lambda > 0$ then (2.6) implies that $\mathrm{Re}\,(h^2(z)) = \frac{1-\lambda^2}{\lambda^2}|h^2(z)|$ for almost every $z \in \mathbb{U}$. Therefore, by the continuity of the functions h^2 and $\mathrm{Re}\,(h^2)$ on the disk $\mathbb{U}$, it follows that $\mathrm{Re}\,(h^2(z)) = \frac{1-\lambda^2}{\lambda^2}|h^2(z)|$ for each $z \in \mathbb{U}$. The last equality yields

$$\Delta|h^2(z)| = 4|h'(z)|^2 = 0$$

and hence

$$\mathrm{Re}\,(h^2(z)) = 0.$$

Thus, the function h is constant on $\mathbb{U}$. Since $h(0) = 0$, we obtain $h \equiv 0$ on $\mathbb{U}$. This contradiction completes the proof. □

3. Proof of the main result

Proof of Theorem 1.1. For simplicity, in this proof we shall often write h instead of $h(z)$. Since the unit disk is simply connected, we have the representation $f = g + \overline{h}$, where g and h are holomorphic functions on the unit disk $\mathbb{U}$ such that $h(0) = 0$. Direct calculations yield

$$|f|^4 = |g|^4 + |h|^4 + 4|g|^2|h|^2 + 4(|g|^2 + |h|^2)\mathrm{Re}(hg) + 2\mathrm{Re}((hg)^2). \tag{3.1}$$

Suppose $g(z) = \sum_{n=0}^{\infty} a_n z^n$ and $h(z) = \sum_{n=1}^{\infty} b_n z^n$ be the Taylor expansions on $\mathbb{U}$ of functions g and h, respectively. Since $f \in h^2$, we have

$$\begin{aligned} \|f\|_{h^2}^4 = \left(\int_{\mathbb{T}} |f(e^{it})|^2 \,\frac{dt}{2\pi}\right)^2 &= \left(\sum_{n=0}^{\infty} |a_n|^2 + \sum_{n=1}^{\infty} |b_n|^2\right)^2 \\ &= \left(\int_{\mathbb{T}} |g(e^{it})|^2 \,\frac{dt}{2\pi} + \int_{\mathbb{T}} |h(e^{it})|^2 \,\frac{dt}{2\pi}\right)^2 \\ &= (\|g\|_{h^2}^2 + \|h\|_{h^2}^2)^2. \end{aligned} \tag{3.2}$$

Thus the functions h and g belong to the Hardy space H^2 and according to (1.5), g and h are also in A^4. From this and the identity $gh = ((g+h)^2 - (g-h)^2)/4$ we see that gh is in A_0^2. Therefore, the all terms on the right of (3.1) are integrable on $\mathbb{U}$. Therefore, we have $f \in b^4$, or equivalently

$$\|f\|_{b^4}^4 = \int_{\mathbb{U}} |f(z)|^4 \, d\sigma < +\infty.$$

By applying the inequality (1.4) to the functions $g^2, h^2 \in H^1$, we immediately obtain

$$\int_{\mathbb{U}} |g(z)|^4 \, d\sigma = \|g^2\|_{A^2}^2 \le \|g^2\|_{H^1}^2, \tag{3.3}$$

$$\int_{\mathbb{U}} |h(z)|^4\, d\sigma = \|h^2\|_{A^2}^2 \le \|h^2\|_{H^1}^2. \tag{3.4}$$

Since $gh \in A^2$, Cauchy-Schwartz inequality together with inequalities (3.3) and (3.4) yields

$$\begin{aligned} \int_{\mathbb{U}} |g(z)|^2|h(z)|^2\, d\sigma &\le \sqrt{\int_{\mathbb{U}} |g(z)|^4\, d\sigma} \cdot \sqrt{\int_{\mathbb{U}} |h(z)|^4\, d\sigma} \\ &= \|g^2\|_{A^2} \cdot \|h^2\|_{A^2} \\ &\le \|g^2\|_{H^1} \cdot \|h^2\|_{H^1} \\ &= \|g\|_{H^2}^2 \cdot \|h\|_{H^2}^2. \end{aligned} \tag{3.5}$$

By using the facts that $h(0)g(0) = 0$, $gh \in A_0^2$, Lemma 2.3 to the holomorphic function gh, and Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} &|\int_{\mathbb{U}} (|h(z)|^2 + |g(z)|^2)\mathrm{Re}\,(h(z)g(z))\, d\sigma| \\ &\le |\int_{\mathbb{U}} (|h(z)|^2 + |g(z)|^2)^2\, d\sigma|^{1/2} |\int_{\mathbb{U}} (\mathrm{Re}\,(h(z)g(z))^2\, d\sigma|^{1/2} \\ &= (\int_{\mathbb{U}} (|h(z)|^2 + |g(z)|^2)^2\, d\sigma)^{1/2} (\int_{\mathbb{U}} \frac{1}{2}|h(z)|^2|g(z)|^2\, d\sigma)^{1/2} \\ &\le (\int_{\mathbb{U}} (|h(z)|^2 + |g(z)|^2)^2\, d\sigma)^{1/2} (\int_{\mathbb{U}} \frac{1}{8}(|h(z)|^2 + |g(z)|^2)^2\, d\sigma)^{1/2} \\ &= \frac{\sqrt{2}}{4} \int_{\mathbb{U}} (|h(z)|^2 + |g(z)|^2)^2\, d\sigma \\ &= \frac{\sqrt{2}}{4} (\|h\|_{H^4}^4 + 2\|hg\|_{H^2}^2 + \|g\|_{H^4}^4). \end{aligned} \tag{3.6}$$

Furthermore, by Lemma 2.2,

$$\int_{\mathbb{U}} \mathrm{Re}\,(h^2(z)g^2(z))\, d\sigma = 0. \tag{3.7}$$

Finally, the relations (3.1)–(3.7), immediately give

$$\begin{aligned} \|f\|_{b^4}^4 &\le (1 + \sqrt{2})(||g||_{H^2}^4 + ||h||_{H^2}^4) + 2(2 + \sqrt{2})||g||_{H^2}^2||h||_{H^2}^2 \\ &\le \frac{3 + 2\sqrt{2}}{2}(||g||_{H^2}^4 + 2||g||_{H^2}^2||h||_{H^2}^2 + ||h||_{H^2}^4) \\ &= \frac{3 + 2\sqrt{2}}{2}\|f\|_{h^2}^4. \end{aligned}$$

The equality in the last inequality of (3.6) is attained if and only if $g = h$ almost everywhere on $\mathbb{U}$. Thus if the equality in (1.5) is attained, then must be $g = h$. Further, the equality in (2.4) is attained if and only if $g^2 \equiv 0$ on $\mathbb{U}$. This means that we have strict inequality in (1.5).

□

References

[1] S. Axler, P. Bourdon, W. Ramey: *Harmonic function theory*, Springer Verlag New York 1992.

[2] J. Burbea: *Sharp inequalities for holomorphic functions.* Illinois J. Math. 31 (1987), 248–264.

[3] T. Carleman: *Zur Theorie der Minimalflächen.* (German) Math. Z. 9 (1921), no. 1-2, 154–160.

[4] F. Hang, X. Wang, X. Yan: *Sharp integral inequalities for harmonic functions.* Comm. Pure Appl. Math. 61 (2008), no. 1, 54–95.

[5] H. Hedenmalm, B. Korenblum, K. Zhu: *Theory of Bergman spaces*, Springer-Verlag New York, 2000.

[6] M. Mateljević, M. Pavlović: *New proofs of the isoperimetric inequality and some generalizations.* J. Math. Anal. Appl. 98 (1984), no. 1, 25–30.

[7] K. Strebel: *Quadratic differentials. Ergebnisse der Mathematik und ihrer Grenzgebiete* (3), 5. Springer-Verlag, Berlin, 1984. xii+184 pp.

Faculty of Natural Sciences and Mathematics, University of Montenegro, Džordža Vašingtona b.b. 81000, Podgorica, Montenegro

E-mail address: davidk@ac.me

Maritime Faculty, University of Montenegro, Dobrota 36, 85330 Kotor, Montenegro

E-mail address: romeo@ac.me