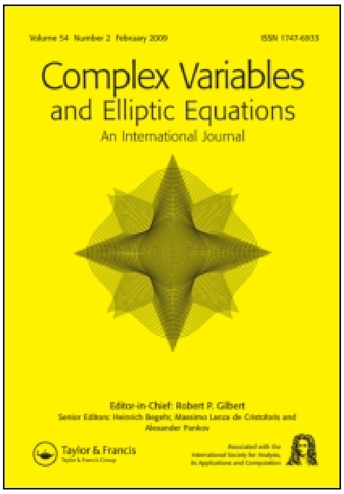

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On Harmonic Diffeomorphisms of the Unit Disc onto a Convex Domain

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We prove a theorem for harmonic diffeomorphisms between the unit disc and a convex Jordan domain, which is a generalization of Heinz theorem [E. Heinz (1959). On one-to-one harmonic mappings. *Pacific J. Math.*, **9**, 101–105] for harmonic diffeomorphisms of the unit disc onto itself. We give a number of corollaries of the theorem we prove.

Keywords: Complex functions; Planar harmonic mappings; Diffeomorphism

1991 Mathematics Subject Classifications: Primary 30C55; Secondary 31A05

1. INTRODUCTION AND AUXILIARY RESULTS

A complex valued function $w = u + iv$, defined in a domain $\Omega \subset \mathbf{C}$, is called a harmonic function if u and v are real valued harmonic functions. If Ω is simply-connected, then there are analytic functions g and h defined on Ω such that w has the representation

$$w = g + \overline{h} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n \overline{z}^n.$$

If w is a harmonic univalent function, then by Lewy's Theorem [10], w has a non-vanishing Jacobian and consequently, according to the inverse mapping theorem, w is a diffeomorphism.

Let w be a sense preserving harmonic diffeomorphism. Then the function

$$a(z) = \frac{h'(z)}{g'(z)}$$

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is called the dilatation of the harmonic function w. Observe that a is an analytic function satisfying the inequality $|a(z)| < 1$. If there exists $k < 1$ such that $|a(z)| < k$ on Ω, then we say that w is a quasiconformal function. We denote by QCH the family of harmonic quasiconformal functions.

In this article we will study the function

$$D(w)(z) = |w_z(z)|^2 + |w_{\bar{z}}(z)|^2.$$

This function is square of the norm of the first differential, and it coincides with the square of the modulus of the complex derivative if the function w is analytic. Heinz [8] proved that if w is a harmonic diffeomorphism of the unit disk onto itself satisfying the condition $w(0) = 0$ then:

$$D(w) \geq \frac{1}{\pi^2}.$$

We shall generalize the result of Heinz under the assumption that the domain of w is the unit disk and the range of w is an arbitrary convex domain.

Let

$$P(r, \theta - \varphi) = \frac{1 - r^2}{2\pi(1 - 2r\cos(\theta - \varphi) + r^2)}$$

denote the Poisson kernel. Then every bounded harmonic function w defined in the unit disc has the representation

$$w(z) = P[g](z) = \int_0^{2\pi} P(r, \theta - \varphi) g(e^{i\theta})\, d\theta, \tag{1.1}$$

where g is a bounded integrable function defined on the unit circle.

Throughout this article Ω denotes a convex domain containing 0, and γ denotes its boundary. Next U denotes the unit disc and S^1 denotes the unit circle. We now state a well-known theorem which plays an important role in the sequel.

Proposition 1.1 (Choquet–Rado–Kneser) [4] *Let γ be a convex Jordan curve in* **C**. *Let g be a homeomorphism from the unit circle S^1 onto the convex Jordan curve γ. Then the function $w(z) = P[g](z)$ is a harmonic diffeomorphism of the unit disc U onto the Jordan domain* $\operatorname{int}\gamma$.

Let $\gamma = \partial\Omega$ be a smooth convex Jordan curve in **C** such that $0 \in \Omega$. We will establish some properties of γ. Let $\varphi \to r(\varphi)e^{i\varphi}$ be the polar parametrization of γ. The tangent t_φ at $\zeta = r(\varphi)e^{i\varphi}$ is defined by

$$y = r(\varphi)e^{i\varphi} + (r'(\varphi) + ir(\varphi))e^{i\varphi}(x - r(\varphi)e^{i\varphi}).$$

The angle α_φ between ζ and the positive oriented tangent at ζ is defined by

$$\cos\alpha_\varphi = \frac{\langle r(\varphi)e^{i\varphi}, (r'(\varphi) + ir(\varphi))e^{i\varphi}\rangle}{r(\varphi)\sqrt{r^2(\varphi) + r'^2(\varphi)}} = \frac{r'(\varphi)}{\sqrt{r^2(\varphi) + r'^2(\varphi)}}. \tag{1.2}$$

Hence

$$\sin\alpha_\varphi = \frac{r(\varphi)}{\sqrt{r^2(\varphi) + r'^2(\varphi)}}.$$

Consequently

$$\cot\alpha_\varphi = \frac{r'(\varphi)}{r(\varphi)}. \tag{1.3}$$

Let $d_\varphi = \operatorname{dist}(t_\varphi, 0)$ be the distance of t_φ from the origin. Then

$$d_\varphi = r(\varphi)\sin\alpha_\varphi. \tag{1.4}$$

Let n_φ be the normal line to the line t_φ that passes through the origin. Since γ is a convex curve, it follows that γ lies to the left of the positive oriented tangent. Hence n_φ cuts γ at some point $\rho(\beta_\varphi)\exp(i\beta_\varphi)$ that lies between t_φ and the origin. Thus, we have

$$r(\varphi) \geq d_\varphi \geq r(\beta_\varphi). \tag{1.5}$$

Let φ_γ be defined by

$$\rho_\gamma = \operatorname{dist}(\gamma, 0) = \min_{z\in\gamma} |z| = r(\varphi_\gamma).$$

Then by (1.5) it follows that $\beta_{\varphi_\gamma} = \varphi_\gamma$. Thus, we deduce the following theorem.

THEOREM 1.2 *Let $\gamma = \partial\Omega$ be a convex Jordan curve in $\mathbf{C}$ such that $0 \in \Omega$. Let $\varphi \to r(\varphi)e^{i\varphi}$ be the polar parametrization of the convex curve γ. Let $d_\varphi = \operatorname{dist}(t_\varphi, 0)$ be the distance of t_φ from the origin. Then there is $\varphi_\gamma \in [0, 2\pi)$ such that*

$$d_\varphi \geq d_{\varphi_\gamma} = r(\varphi_\gamma) = \operatorname{dist}(\gamma, 0) \tag{1.6}$$

for all $\varphi \in [0, 2\pi)$.

Let $g: S^1 \to \gamma$ be a continuous locally injective function from the unit circle S^1 onto the convex Jordan curve γ. Then

$$F(\varphi) = \rho(\varphi)e^{if(\varphi)} = g(e^{i\varphi}), \quad \varphi \in [0, 2\pi)$$

is a parametrization of γ which represents g. If g is a orientation preserving then f obviously is monotone increasing. Suppose that F is differentiable. Let $\varphi \to r(\varphi)e^{i\varphi}$

be the polar parametrization of γ. Since $r(f(\varphi)) = \rho(\varphi)$, we deduce that $\rho'(\varphi) = r'(f(\varphi)) \cdot f'(\varphi)$. Hence

$$r'(f(\varphi)) = \frac{\rho'(\varphi)}{f'(\varphi)}. \tag{1.7}$$

The following lemma gives an important property of convex curves.

LEMMA 1.3 *Let γ be a convex Jordan curve in* $\mathbf{C}$. *Let*

$$[0, 2\pi) \ni \varphi \to F(\varphi) = \rho(\varphi)e^{if(\varphi)} \in \gamma$$

be a locally injective differentiable parametrization of γ. Then

$$\begin{aligned} K(x, \varphi) = \rho^2(\varphi)f'(\varphi) - \rho'(\varphi)\rho(x)\sin(f(\varphi) - f(x)) \\ - \rho(\varphi)\rho(x)f'(\varphi)\cos(f(\varphi) - f(x)) \geq 0 \end{aligned} \tag{1.8}$$

for all $\varphi, x \in [0, 2\pi)$.

Proof Let $\zeta = \rho(\varphi)e^{if(\varphi)}$ and let $y = \rho(x)e^{if(x)}$. Now, let $\mathbf{n}_\zeta$ be the outer normal of the curve γ at ζ. Since the function f is monotone increasing, it follows that

$$\mathbf{n}_\zeta = -i \cdot \zeta_\varphi(\varphi) = -i \cdot (\rho'(\varphi) + i\rho(\varphi)f'(\varphi))e^{if(\varphi)}.$$

Since γ is convex, it follows that

$$\langle \zeta - y, \mathbf{n}_\zeta \rangle \geq 0.$$

Then the inequality of the lemma easily follows. ■

The inequality (1.8) will be used in the proof of our main theorem (see Theorem 2.2).

2. THE MAIN RESULTS

Throughout this section, we will use the notations

$$w_z(e^{i\varphi}) := \lim_{r \to 1} w_z(re^{i\varphi}) \quad \text{and} \quad w_{\bar{z}}(e^{i\varphi}) := \lim_{r \to 1} w_{\bar{z}}(re^{i\varphi})$$

if the limits exist.

LEMMA 2.1 *Let $w = u + iv$ be a differentiable function defined in a domain $\Omega \subset \mathbf{C}$. Then*:

$$J_w(re^{i\varphi}) = u_x v_y - u_y v_x = |w_z|^2 - |w_{\bar{z}}|^2 = \frac{1}{r}(u_r v_\varphi - u_\varphi v_r),$$

and

$$D(w) = |w_z|^2 + |w_{\bar{z}}|^2 = \frac{|w_r|^2}{2} + \frac{|w_\varphi|^2}{2r^2}.$$

Proof From

$$w_r = e^{i\varphi} w_z + e^{-i\varphi} w_{\bar z} \quad \text{and} \quad w_\varphi = ir(e^{i\varphi} w_z - e^{-i\varphi} w_{\bar z}),$$

we have

$$w_z = e^{-i\varphi}\Big(w_r - i\frac{w_\varphi}{r}\Big) \quad \text{and} \quad w_{\bar z} = e^{i\varphi}\Big(w_r + i\frac{w_\varphi}{r}\Big).$$

Hence,

$$|w_z|^2 - |w_{\bar z}|^2 = \frac{1}{r}(u_r v_\varphi - u_\varphi v_r),$$

and

$$|w_z|^2 + |w_{\bar z}|^2 = \frac{|w_r|^2}{2} + \frac{|w_\varphi|^2}{2r^2}. \qquad \blacksquare$$

We are now ready to state the main result of this article.

THEOREM 2.2 *Let $\gamma = \partial\Omega$ be a convex Jordan curve in $\mathbf{C}$ such that $0 \in \Omega$. Let $g: S^1 \to \gamma$ be a C^2 homeomorphism of the unit circle onto γ. Let $w(z) = P[g](z)$ and let $w(0) = 0$. If $F(\varphi) = \rho(\varphi)e^{if(\varphi)} = g(e^{i\varphi})$, then*

$$\lim_{r\to 1} D(w)(re^{i\varphi}) \geq \frac{|F'(\varphi)|^2}{2} + \frac{1}{8}\rho_\gamma^2 \tag{2.1}$$

for all $\varphi \in [0, 2\pi)$ and

$$D(w)(z) \geq \tfrac{1}{16}\rho_\gamma^2 \tag{2.2}$$

for all $z \in U$. Here ρ_γ denotes $\min_{z\in\gamma}|z|$.

Proof Because of Lemma 2.1, we have

$$D(w) = \frac{1}{2}\left(u_r^2 + v_r^2 + \frac{u_\varphi^2 + v_\varphi^2}{r^2}\right). \tag{2.3}$$

On the other hand, the assumption $F \in C^2$ implies that w_φ and w_r have continuous extensions to the boundary. (See [6,9].)

Hence, the following limit relations hold:

$$\lim_{r\to 1} u_\varphi(re^{i\varphi}) = u_\varphi(e^{i\varphi}) = \rho'(\varphi)\cos f(\varphi) - \rho(\varphi) f'(\varphi)\sin f(\varphi), \tag{2.4}$$

$$\lim_{r\to 1} v_\varphi(re^{i\varphi}) = v_\varphi(e^{i\varphi}) = \rho'(\varphi)\sin f(\varphi) + \rho(\varphi) f'(\varphi)\cos f(\varphi), \tag{2.5}$$

$$\lim_{\rho\to 1} u_r(\rho e^{i\varphi}) = \lim_{r\to 1} \frac{u(re^{i\varphi}) - u(e^{i\varphi})}{r-1}, \tag{2.6}$$

$$\lim_{\rho\to 1} v_r(\rho e^{i\varphi}) = \lim_{r\to 1} \frac{v(re^{i\varphi}) - v(e^{i\varphi})}{r-1}. \tag{2.7}$$

By exploiting the representation (1.1), (2.3)–(2.7) and Fubini's theorem, we have:

$$\begin{aligned}
\lim_{r\to 1} D(w)(re^{i\varphi}) &= \frac{1}{2}\lim_{r\to 1}\left(u_r^2 + v_r^2 + \frac{u_\varphi^2 + v_\varphi^2}{r^2}\right)\\
&= \tfrac{1}{2}(\rho'^2(\varphi) + \rho^2(\varphi) f'^2(\varphi)) + \tfrac{1}{2}\lim_{r\to 1}(u_r^2 + v_r^2)\\
&= L + \frac{1}{2}\lim_{r\to 1}\int_0^{2\pi}\int_0^{2\pi}[\rho(\varphi)\cos f(\varphi) - \rho(x)\cos f(x)]\\
&\quad\times[\rho(\varphi)\cos f(\varphi) - \rho(y)\cos f(y)]\frac{P(r, x-\varphi)P(r, y-\varphi)}{(1-r)^2}\,dx\,dy\\
&\quad+\frac{1}{2}\lim_{r\to 1}\int_0^{2\pi}\int_0^{2\pi}[\rho(\varphi)\sin f(\varphi) - \rho(x)\sin f(x)]\\
&\quad\times[\rho(\varphi)\sin f(\varphi) - \rho(y)\sin f(y)]\frac{P(r, x-\varphi)P(r, y-\varphi)}{(1-r)^2}\,dx\,dy\\
&= L + \frac{1}{2}\lim_{r\to 1}\int_0^{2\pi}\int_0^{2\pi}[\rho(\varphi) - \rho(x)\cos(f(\varphi) - f(x))]\\
&\quad\times[\rho(\varphi) - \rho(y)\cos(f(\varphi) - f(y))]\frac{P(r, x-\varphi)P(r, y-\varphi)}{(1-r)^2}\,dx\,dy\\
&\quad+\frac{1}{2}\lim_{r\to 1}\int_0^{2\pi}\int_0^{2\pi}\rho(x)\sin(f(\varphi) - f(x))\\
&\quad\times\rho(y)\sin(f(\varphi) - f(y))\frac{P(r, x-\varphi)P(r, y-\varphi)}{(1-r)^2}\,dx\,dy\\
&= L + \frac{1}{2}\lim_{r\to 1}\left(\int_0^{2\pi}[\rho(\varphi) - \rho(x)\cos(f(\varphi) - f(x))]\frac{P(r, \varphi - x)}{1-r}\,dx\right)^2\\
&\quad+\frac{1}{2}\left(\lim_{r\to 1}\int_0^{2\pi}\rho(x)\sin(f(\varphi) - f(x))\frac{P(r, \varphi - x)}{1-r}\,dx\right)^2\\
&= L + \tfrac{1}{2}\lim_{r\to 1}((a + \beta b)^2 + b^2),
\end{aligned}$$

where

$$\begin{aligned}
a &= \int_0^{2\pi}\left(\rho(\varphi) - \rho(x)\cos(f(\varphi) - f(x)) - \frac{\rho'(\varphi)\rho(x)}{\rho(\varphi) f'(\varphi)}\sin(f(\varphi) - f(x))\right)\\
&\quad\times\frac{P(r, \varphi - x)}{1-r}\,dx,
\end{aligned}$$

$$\beta = \frac{\rho'(\varphi)}{f'(\varphi)\rho(\varphi)}, \qquad L = \tfrac{1}{2}\big(\rho'(\varphi)^2 + \rho^2(\varphi)f'^2(\varphi)\big),$$

and

$$b = \int_0^{2\pi} \rho(x)\sin(f(\varphi) - f(x))\frac{P(r, \varphi - x)}{1 - r}\,dx.$$

Then we have

$$\begin{aligned}(a + \beta b)^2 + b^2 &= a^2 + \beta^2 b^2 + 2\beta ab + b^2 \\ &= a^2 + (\beta^2 + 1)\left(b^2 + \frac{2\beta ab}{1 + \beta^2}\right) \\ &= a^2 + \left(b + \frac{\beta a}{1 + \beta^2}\right)^2(\beta^2 + 1) - \frac{\beta^2 a^2}{1 + \beta^2} \\ &= \frac{1}{1 + \beta^2}a^2 + \left(b + \frac{\beta a}{1 + \beta^2}\right)^2(\beta^2 + 1) \\ &\geq \frac{a^2}{1 + \beta^2} = \frac{1}{1 + \beta^2}\left(\int_0^{2\pi} \frac{K(x, \varphi)}{\rho(\varphi)f'(\varphi)}\frac{P(r, \varphi - x)}{1 - r}\,dx\right)^2.\end{aligned}$$

Here $K(x, \varphi)$ denotes the function defined in (1.8). K is positive because the range $\Omega = \operatorname{int}\gamma$ is convex. Consequently, the integrand in the last integral is positive. On the other hand

$$\frac{P(r, \varphi - x)}{1 - r} \geq \frac{1}{4\pi}.$$

Since $w(0) = 0$, we have

$$\frac{a^2}{1 + \beta^2} \geq \frac{1}{1 + \beta^2}\left(\int_0^{2\pi} \frac{K(x, \varphi)}{\rho(\varphi)f'(\varphi)}\frac{dx}{4\pi}\right)^2 \geq \frac{1}{1 + \beta^2}\left(\int_0^{2\pi} \rho(\varphi)\frac{dx}{4\pi}\right)^2 = \frac{1}{1 + \beta^2}\frac{\rho^2(\varphi)}{4}.$$

Consequently,

$$\lim_{r \to 1} D(w)(re^{i\varphi}) \geq \frac{1}{2}(\rho'^2(\varphi) + \rho^2(\varphi)f'^2(\varphi)) + \frac{1}{8}\frac{\rho^2(\varphi)}{1 + (\rho'(\varphi)/f'(\varphi)\rho(\varphi))^2}.$$

Since the function $\varphi \to F(\varphi)$ is differentiable, Eqs. (1.3) and (1.7) imply that

$$\left(\frac{\rho'(\varphi)}{f'(\varphi)\rho(\varphi)}\right)^2 = \cot^2\alpha_{f(\varphi)},$$

where $\alpha_{f(\varphi)}$ has been defined in (1.2).

Then by Theorem 1.2 and (1.4), it follows that

$$D(w)(e^{i\varphi}) \geq \frac{|F'(\varphi)|^2}{2} + \frac{\rho^2(\varphi)}{8(1+\cot^2\alpha_{f(\varphi)})}$$

$$= \frac{|F'(\varphi)|^2}{2} + \frac{r^2(f(\varphi))\sin^2\alpha_{f(\varphi)}}{8} \geq \frac{|F'(\varphi)|^2}{2} + \frac{1}{8}\rho_\gamma^2.$$

Thus we have proved (2.1). We now turn to the proof of (2.2). Since

$$\frac{\partial}{\partial\varphi}w(z) = izw_z(z) - i\,\bar{z}w_{\bar{z}}(z) = P[F'](z),$$

it follows that $|F'(\varphi)|^2 \geq (|w_z(e^{i\varphi})| - |w_{\bar{z}}(e^{i\varphi})|)^2$. Then by inequality (2.1), we obtain

$$|w_z(e^{\varphi})|^2 + |w_{\bar{z}}(e^{\varphi})|^2 + 2|w_z(e^{\varphi})w_{\bar{z}}(e^{\varphi})| \geq \tfrac{1}{4}\rho_\gamma^2.$$

Hence

$$|w_z(e^{\varphi})| + |w_{\bar{z}}(e^{\varphi})| \geq \tfrac{1}{2}\rho_\gamma.$$

Since g is a homeomorphism, Proposition 1.1, implies that w is a univalent harmonic function. According to Lewy's theorem, we conclude that the Jacobian of the harmonic function w is positive on the unit disc. Consequently

$$|w_z(e^{i\varphi})| \geq \tfrac{1}{4}\rho_\gamma.$$

Since the analytic function w_z is non-vanishing on U, we have

$$|w_z(re^{i\varphi})| \geq \tfrac{1}{4}\rho_\gamma,$$

by the minimum principle. Finally, we deduce that

$$D(w)(z) \geq \tfrac{1}{16}\rho_\gamma^2,$$

for every $z \in U$ and thus the proof is complete. ■

Remark 2.3 In the proof of the first inequality of Theorem 2.2 we have used the fact that F is a differentiable parametrization of the Jordan curve γ satisfying condition

$$K(x,\varphi) = \rho^2(\varphi)f'(\varphi) - \frac{\partial}{\partial\varphi}(\rho(\varphi)\rho(x)\sin(f(\varphi)-f(x))) \geq 0.$$

We note that the last inequality holds by virtue of local injectivity of F. Thus by following the proof of Theorem 2.2, we may deduce the following theorem.

THEOREM 2.4 *Let $\gamma = \partial\Omega$ be a convex Jordan curve in* $\mathbf{C}$ *such that* $0 \in \Omega$. *Let* $F(\varphi) = \rho(\varphi)e^{if(\varphi)}$ *be a* C^2 *locally injective function from* $[0, 2\pi)$ *onto* γ. *Next, let* $w = P[F](z)$ *be the Poisson integral of the function* F. *Then*:

$$D(w)(e^{i\varphi}) = \lim_{r\to 1} D(w)(re^{i\varphi}) \geq \frac{|F'(\varphi)|^2}{2} + \frac{1}{8}\rho_\gamma^2$$

where $\rho_\gamma = \min_{z\in\gamma}|z|$.

We note that in Theorem 2.4, w is not necessarily univalent.

The question arises whether the main theorem holds for an arbitrary harmonic diffeomorphism. The answer to this question is positive. Indeed, the next theorem holds.

THEOREM 2.5 *Let Ω be a convex domain in* $\mathbf{C}$ *containing the origin. Let* $w\colon U \to \Omega$ *be a harmonic diffeomorphism of the unit disc onto* Ω. If $w(0) = 0$, *then*

$$D(w)(z) \geq \tfrac{1}{16}\rho_\gamma^2, \tag{2.8}$$

where $\gamma = \partial\Omega$, $\rho_\gamma = \min_{z\in\gamma}|z|$.

Proof Let $h\colon U \to \Omega$ be the Riemann mapping; i.e., the conformal mapping of the unit disc onto the convex domain Ω such that $h(0) = 0$ and $h'(0) > 0$. Since Ω is convex we have

$$\mathrm{Re}\left(\frac{zh''(z)}{h'(z)}\right) \geq -1.$$

Next, let $h_r(z) = h(rz)$. Then

$$\frac{zh_r''(z)}{h_r'(z)} = \frac{zrh''(zr)}{h'(zr)}.$$

Hence

$$\mathrm{Re}\left(\frac{zh_r''(z)}{h_r'(z)}\right) \geq -1.$$

Thus, we conclude that $h(rS^1)$ is a convex analytic curve on Ω for every $r \in (0, 1)$. Let $\Delta_n = h((n/n+1)U)$, and $D_n = w^{-1}(\Delta_n)$. Let g_n be a Riemann mapping from the unit disc onto D_n (we may suppose that $0 \in D_n$ for large enough n). Let $w_n = w \circ g_n$. Then w_n is a harmonic diffeomorphism from the unit disc onto the convex Jordan domain Δ_n. Let $\gamma_n = \partial\Delta_n$. By applying Theorem 2.2, we conclude

$$D(w_n)(z) \geq \tfrac{1}{16}\rho_{\gamma_n}^2, \tag{2.9}$$

because w and $g_n, n \in \mathbf{N}$ have C^∞ differentiable extensions to the boundary (see for example [3]).

By applying the theorem on normal families of analytic functions, we see that there exists a subsequence (g_{n_k}) of the sequence (g_n) which converges uniformly on every compact set to an analytic function g such that $g(0) = 0$. Since

$$\left\{w\colon |w| \le 1 - \frac{1}{n_k}\right\} \subset D_{n_k} \subset \{w\colon |w| < 1\}$$

Schwarz's lemma implies that

$$\left(1 - \frac{1}{n_k}\right)|z| \le |g_{n_k}(z)| < |z|.$$

Consequently, g is an analytic univalent function from the unit disc onto itself satisfying the conditions $g(0) = 0$ and $g'(0) > 0$. Therefore, $g = Id$. Consequently, the sequence g'_{n_k} converges uniformly to the function $g'(z) = 1$ on every compact set. On the other hand, the sequence ρ_{γ_n} converges to ρ_γ. By the inequality (2.9) and by the equality

$$D(w_{n_k}) = |g'_{n_k}(z)|^2 D(w \circ g_{n_k}),$$

we obtain

$$D(w)(z) = \lim_{k\to\infty} D(w_{n_k})(z) \ge \tfrac{1}{16}\rho_\gamma^2$$

for every $z\colon |z| < 1$. ■

Remark 2.6 The conditions $w(0) = 0$ and $0 \in \operatorname{int}\gamma$ can be omitted. Indeed, by setting $w_1(z) = w(z) - w(0)$, the problem is reduced to the previous case.

We now estimate the function $D(w)$ under the assumption that w is a harmonic quasiconformal mapping. We do so by means of the following.

COROLLARY 2.7 *Let Ω be a convex domain containing the origin. Let w be a harmonic k-quasiconformal function from the unit disc U onto Ω, such that $w(0) = 0$. Then*

$$D(w)(z) \ge \frac{1}{4(1 + 2k + k^2)}\rho_\gamma^2, \tag{2.10}$$

where ρ_γ is the distance of $\gamma = \partial\Omega$ from the origin.

Proof We divide the proof in two steps.

Step 1 Assume that $w = P[F]$, where F is a twice continuously differentiable function. Then by Theorem 2.5, we have

$$D(w)(e^{i\varphi}) \ge \frac{|F'(\varphi)|^2}{2} + \frac{1}{8}\rho_\gamma^2.$$

The function

$$\frac{\partial}{\partial\varphi} w(z) = i(zw_z(z) - \bar{z}w_{\bar{z}}(z)) = P[F'](z), \quad (z = re^{i\varphi}),$$

has a continuous extension to the boundary. Hence, we have:

$$|F'(\varphi)|^2 = \lim_{r\to 1} \left| \frac{\partial}{\partial\varphi} w(z) \right|^2 \geq (|w_z(e^{i\varphi})| - |w_{\bar{z}}(e^{i\varphi})|)^2.$$

As in the proof of Theorem 2.2, we obtain $|w_z(e^{i\varphi})| + |w_{\bar{z}}(e^{i\varphi})| \geq (1/2)\rho_\gamma$. Since w is k-quasiconformal, we also have

$$\lim_{r\to 1} |w_z(re^{i\varphi})| \geq \frac{1}{2(1+k)}\rho_\gamma.$$

Since w_z is a non-vanishing analytic function we deduce that

$$|w_z(z)| \geq \frac{1}{2(1+k)}\rho_\gamma \tag{2.11}$$

for all $z \in U$.

Step 2 Let w be an arbitrary k-quasiconformal harmonic mapping from the unit disc onto the domain Ω. Then as in the proof of Theorem 2.5, we construct the sequence $\{w_n\}$ of harmonic diffeomorphisms, which have twice continuously differentiable extensions to the boundary and which are k-quasiconformal mappings. The functions w_n are k-quasiconformal because w is k-quasiconformal mapping. Thus, the sequence $\{w_n\}$ converges uniformly to the function w on every compact set, and the sequence ρ_{γ_n} tends to ρ_γ. On the other hand, the sequence of the analytic functions w_{nz} converges uniformly to w_z on every compact set. Hence the inequality (2.11) holds for an arbitrary k-quasiconformal harmonic mapping w. From inequality (2.11) it follows that

$$D(w)(z) \geq \frac{1}{4(1+2k+k^2)}\rho_\gamma^2,$$

which yields the conclusion. ■

Observe that in the case of the unit disk, the last inequality of the previous proof has the form

$$D(w)(z) \geq \frac{1}{4(1+2k+k^2)}.$$

This inequality is better than the inequality of Heinz if k is close to 0.

Corollary 2.8 *Let Ω be a convex domain in $\mathbf{C}$ containing the origin. Let w be a conformal mapping of the unit disc onto Ω such that $w(0) = 0$. Then for all $z \in U$, we have*

$$|w'(z)| > \rho_\gamma/2, \tag{2.12}$$

where ρ_γ is the distance of $\gamma = \partial\Omega$ from the origin.

The following statement is an immediate corollary of the previous statement.

Corollary 2.9 *Let Ω be a convex domain in $\mathbf{C}$ containing the origin. Let w be a conformal mapping of Ω onto the unit disc U such that $w(0) = 0$. Then*

$$|w'(z)| < 2/\rho_\gamma,$$

where ρ_γ is defined above.

The following statement follows at once from Theorem 2.5.

Corollary 2.10 *Let*

$$w = h(z) + \overline{g(z)} = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n$$

be a harmonic diffeomorphism of the unit disc onto a convex domain $\Omega \subset \mathbb{C}$. Then

$$|a_1|^2 + |b_1|^2 \geq \tfrac{1}{16}\rho^2, \tag{2.13}$$

where $\rho = \operatorname{dist}(\partial\Omega, 0)$.

The following example shows that the condition of convexity is important.

Example 2.11 The function $w(z) = (z+1)^2 - 1$ is a conformal mapping between the unit disc and the Jordan domain $w(U)$ which is not convex, and which satisfies $w'(-1) = 0$. Hence, the inequality (2.12) does not hold for non-convex domains.

The next example shows that the inequality (2.12) is sharp.

Example 2.12 Let $n \in \mathbf{N}$. Then the function w_n defined by

$$w_n(z) = \frac{(n+1)z + n}{n+1+nz} - \frac{n}{n+1}$$

is a conformal mapping between the unit disc U and the disc

$$U_n = U - \frac{n}{n+1} = \operatorname{int}\gamma_n$$

and satisfies the conditions

$$w_n(0) = 0 \quad \text{and} \quad w'_n(1) = \frac{1}{2n+1} \geq \frac{1}{2}\rho_{\gamma_n} = \frac{1}{2(n+1)}.$$

Moreover, the mapping $w(z) = \lim_{n\to\infty}(n+1)w_n(z) = 2z/1+z$ is a conformal mapping of the unit disk onto the half-plane $x < 1$ with $w'(1) = 1/2$. Thus the constant 1/2 in (2.12) is the best possible.

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