

# ON CERTAIN NONLINEAR ELLIPTIC PDE AND QUASICONFORMAL MAPS BETWEEN EUCLIDEAN SURFACES

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ABSTRACT. We mainly investigate some properties of quasiconformal mappings between smooth 2-dimensional surfaces with boundary in the Euclidean space, satisfying certain differential equations (inequalities) concerning Laplacian, and in particular satisfying Laplace equation and show that that these mappings are Lipschitz. Conformal parametrization of such surfaces and the method developed in our paper [7] have important role in this paper.

## 1. INTRODUCTION

By  $\mathbb{U}$  we denote the unit disk, by  $\Omega$  a domain in  $\mathbb{R}^2$  and by  $S$  a smooth 2-dimensional surface in  $\mathbb{R}^l$ ,  $l \geq 3$ .

Let  $f$  be a smooth mapping between a Jordan domain  $\Omega$  and a surface  $S$  of the Euclid space  $\mathbb{R}^l$ . Consider the functional

$$(1.1) \quad E[f] = \iint_{\Omega} |f_x|^2 + |f_y|^2 dx dy$$

The stationary points of the energy integral  $E[f]$  satisfy the Euler-Lagrange equation i.e. Laplace equation

$$(1.2) \quad \Delta f = f_{xx} + f_{yy} = 4f_{z\bar{z}} \equiv 0.$$

The mapping  $f$  satisfying the relation (1.2) is called *harmonic*.

Let us define harmonic mappings and quasiconformal mappings between two smooth 2-dimensional surfaces  $S_1 \subset \mathbb{R}^m$  and  $S_2 \subset \mathbb{R}^n$ . For every  $a \in S_1$  let  $X_a(x, y)$  be a *conformal mapping* between the unit disk and a neighborhood  $U_a \subset S_1$  i.e. let  $x, y$  be *isothermal coordinates* in  $U_a$ . The mapping  $f$  of the surface  $S_1$  into the surface  $S_2$  is called *harmonic* if for every  $a \in S_1$   $f \circ X_a : \mathbb{U} \rightarrow \mathbb{R}^n$  is harmonic in  $\mathbb{U}$ . Let  $Y = Y_{f(a)}$  be isothermal coordinates in some neighborhood  $V_{f(a)}$  in  $Y$ . It means that  $g = (Y_{f(a)})^{-1} \circ f \circ X_a$  is  $\rho$ -harmonic, where  $\rho(w) = |Y_u(w)|^2$ ; we also say that  $f$  is harmonic with respect to the metric on  $S_2$  inherited from the Euclid space  $\mathbb{R}^n$ . Let  $k \in [0, 1)$  and let  $f$  be a homeomorphism between  $S_1$  and  $S_2$ . Let  $a \in S_1$  be arbitrary and let  $X_a$  be isothermal coordinates in  $U_a$ . Similarly let  $Y_{f(a)}$  be isothermal coordinates in some neighborhood  $V_{f(a)}$  in  $Y$ . If for every  $a$  the mapping  $g = (Y_{f(a)})^{-1} \circ f \circ X_a$  satisfies the inequality  $|g_{\bar{z}}| \leq k|g_z|$  in  $X_a^{-1}(U_a)$ , then  $f$  is said to be a *k quasiconformal (q.c.) mapping*.

The domain  $\Omega$  (the surface  $S$ ) is called  $C^{l,\alpha}$  domain (surface) if the boundary  $\partial\Omega$  ( $\partial S$ ) is a compact  $C^{l,\alpha}$  1-dimensional manifold (curve).

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In this paper we continue to study the boundary behaviors of q.c. harmonic mappings between plane domains and Euclid surfaces. Notice this important fact, the class of q.c. harmonic mappings contains conformal mappings (see the section below for boundary behaviors of conformal mappings).

Further developments of the method presented in [7] leads to Theorem 3.1 and 3.4: it is proved that every q.c.  $C^2$  diffeomorphism  $w$  between two plane domains with smooth boundary satisfying the inequality

$$(1.3) \quad |\Delta w| \leq M|\nabla w|^2 + N$$

is Lipschitz continuous. The inequality we will call a *Poisson differential inequality*. These theorems imply corresponding results for q.c. harmonic mappings between smooth surfaces (Theorem 3.6, Theorem 3.7). This extends the results of the authors [7] where instead of (1.3) is assumed that

$$(1.4) \quad |\Delta w| \leq M|w_z w_{\bar{z}}|.$$

See [11] for the first result and see [7]-[6], [16] and [15] for some recent results on the topic of q.c. harmonic mappings for plane domains.

## 2. CONFORMAL PARAMETRIZATION

**Proposition 2.1** (Kellogg and Warshawski see [10], [17] and [18]). *If  $\Omega$  is a Jordan domain having a  $C^{l,\alpha}$  boundary and  $\omega$  conformal mapping of  $\Omega$  onto  $D$ , then  $\omega^{(l)} \in C^\alpha(\bar{\Omega})$ . In particular it is bounded from above on  $\mathbb{U}$ .*

The following theorem can be viewed as an extension of Proposition 2.1 and of Riemann mapping theorem.

**Theorem 2.2.** [3, Theorem 3.1] *Suppose  $S$  is a surface with boundary, homeomorphic to a plane domain  $G$  bounded by  $k$  circles via a chart  $\psi : \bar{G} \mapsto S$ . Suppose the coefficients of the metric tensor of  $S$  can be defined in this chart by bounded measurable functions  $g_{ij}$  with  $g_{11}g_{22} - g_{12}^2 \geq \lambda > 0$  in  $G$ . Then  $S$  admits a conformal representation  $\tau \in H_1^2 \cap C^\alpha(\bar{B}, \bar{G})$ , where  $B$  is a plane domain bounded by  $k$  circles and  $\tau$  satisfies almost everywhere the conformality relations*

$$|\tau_x|^2 = |\tau_y|^2 \text{ and } \langle \tau_x, \tau_y \rangle = 0$$

(Here  $(x, y)$  denote the coordinates of points in  $B$ , and norms and products are taken with respect to the metric of  $S$ ).

$\tau$  can be normalized by a three point condition, namely three points on one of the boundary curves of  $S$  can be made to correspond, respectively, to three given points on the outer boundary of  $B$  which can be taken as the unit circle, or by fixing the image of an interior point. Furthermore, concerning higher regularity,  $\tau$  is as regular as  $S$ , i.e. if  $S$  is of class  $C^{m,\alpha}(\bar{B})$  ( $m \in \mathbb{N}$ ,  $0 < \alpha < 1$ ) or in  $C^\infty$  then also  $\tau \in C^{m,\alpha}(\bar{B})$  or  $\tau \in C^\infty(\bar{B})$ , respectively. In particular, if  $S$  is at least  $C^{1,\alpha}$  then the conformality relations are satisfied everywhere, and  $\tau$  is a diffeomorphism.

We will make use the following corollary of the previous theorem.

**Corollary 2.3.** *Let  $X : \mathbb{U} \mapsto S$  be a conformal mapping between the unit disk and a  $C^{2,\alpha}$  surface  $S$ . Then*

$$(2.1) \quad c := \min_{\{z:|z|\leq 1\}} |X_u(z)| = \min_{\{z:|z|\leq 1\}} |X_v(z)| > 0,$$

$$(2.2) \quad C := \max_{\{z:|z|\leq 1\}} |X_{uu}(z)| + |X_{uv}(z)| + |X_{vv}(z)| < \infty, \text{ and}$$

$$(2.3) \quad |\log |X_u(w)|^2_w| \leq M' < \infty.$$

*Proof.* The first two inequalities follows directly from Theorem 2.2. For  $\rho = \log |X_u(w)|^2$  we have

$$\rho_w = \frac{\langle X_{uu}, X_u \rangle - i \langle X_{uv}, X_u \rangle}{|X_u(w)|^2} = \frac{\langle X_{uu}, X_u \rangle + i \langle X_{uu}, X_v \rangle}{|X_u(w)|^2}.$$

Consequently:

$$(2.4) \quad |\rho_w| \leq 2 \frac{|X_{uu}|}{|X_u|} \leq M' = \frac{C}{c}.$$

□

### 3. THE MAIN RESULTS

Firstly we are going to establish a local Lipschitz character of our mappings.

**Theorem 3.1** (The main theorem). *Let  $f$  be a quasiconformal  $C^2$  diffeomorphism from the plane domain  $\Omega$  onto the plane domain  $G$ . Let  $\gamma_\Omega \subset \partial\Omega$  and  $\gamma_G = f(\gamma_\Omega) \subset \partial G$  be  $C^{1,\alpha}$  respectively  $C^{2,\alpha}$  Jordan arcs. If for some  $\tau \in \gamma_\Omega$  there exist positive constants  $r$ ,  $M$  and  $N$  such that*

$$(3.1) \quad |\Delta f| \leq M |\nabla f|^2 + N, \quad z \in \Omega \cap D(\tau, r),$$

*then  $f$  has bounded partial derivatives in  $\Omega \cap D(\tau, r_\tau)$  for some  $r_\tau < r$ . In particular it is a Lipschitz mapping in  $\Omega \cap D(\tau, r_\tau)$ .*

We need the following proposition.

**Proposition 3.2. (Heinz-Bernstein, see [2]).** *Let  $s : \overline{\mathbb{U}} \rightarrow \mathbb{R}$  be a continuous function from the closed unit disc  $\overline{\mathbb{U}}$  into the real line satisfying the conditions:*

- (1)  $s$  is  $C^2$  on  $\mathbb{U}$ ,
- (2)  $s_b(\theta) = s(e^{i\theta})$  is  $C^2$  and
- (3)  $|\Delta s| \leq M_0 |\nabla s|^2 + N_0$ , on  $\mathbb{U}$  for some constants  $M_0$  and  $N_0$ .

*Then the function  $|\nabla s|$  is bounded on  $\mathbb{U}$ .*

*Proof of Theorem 3.1.* Let  $r > 0$  be sufficiently small positive real number such that  $\Delta = D(\tau, r) \cap \Omega$  is a Jordan domain with  $C^{1,\alpha}$  boundary consisting of a circle arc  $C(t_0, t_1)$  and an arc  $\gamma_0[t_0, t_1] \subset \gamma$  containing  $\tau$ . Take  $D = f(\Delta)$ . Let  $g$  be a conformal mapping of the unit disc onto  $\Delta$ . Let  $\tilde{f} = f \circ g$ . Since  $\Delta \tilde{f} = |g'|^2 \Delta f$  and  $|\nabla \tilde{f}|^2 = |g'|^2 |\nabla f|^2$ , we find that  $\tilde{f}$  satisfies the inequality (3.1) with  $M_1 = M$  and  $N_1 = N \cdot \inf_{|z| \leq 1} |g'(z)|^{-1}$ . We will prove the theorem for  $\tilde{f}$  and then apply Kellogg's theorem. For simplicity, we write  $f$  instead of  $\tilde{f}$ . Let  $J$  be a compact subset of  $\gamma_0$  containing  $\tau$  but not containing the points  $t_0$  and  $t_1$ . Let  $t \in J$  be arbitrary.

**Step 1 (Local Construction).** In this step we show that there are two Jordan domains  $D_1$  and  $D_2$  in  $D$  with  $C^{2,\alpha}$  boundary such that

- (i)  $D_1 \subset D_2 \subset D$ ,
- (ii)  $\partial D \cap \partial D_2$  is a connected arc containing the point  $w = f(t)$  in its interior,
- (iii)  $\emptyset \neq \overline{\partial D_2} \setminus \partial D_1 \subset D$ .

Let  $H_1$  be the Jordan domain bounded by the Jordan curve  $\gamma_1$  which is composed by the following sequence of Jordan arcs:  $\{y^{1/5} + (2-x)^{1/5} = 1, 1 \leq x \leq 2\}$ ;  $\{(2-y)^{1/5} + (2-x)^{1/5} = 1, 1 \leq x \leq 2\}$ ;  $[(1, 2), (-1, 2)]$ ;  $\{(2-y)^{1/5} + (2+x)^{1/5} =$

$1, -2 \leq x \leq -1\}$ ;  $\{y^{1/5} + (2+x)^{1/5} = 1, -2 \leq x \leq -1\}$  and  $[(-1, 0), (1, 0)]$ . Let  $H_2$  be the Jordan domain bounded by the Jordan curve  $\gamma_2$  which is composed by the following sequence of Jordan arcs:  $\{y^{1/5} + (2-x)^{1/5} = 1, 1 \leq x \leq 2\}$ ;  $[(2, 1), (2, 2)]$ ;  $\{(3-y)^{1/5} + (2-x)^{1/5} = 1, 1 \leq x \leq 2\}$ ;  $[(1, 3), (-1, 3)]$ ;  $\{(3-y)^{1/5} + (2+x)^{1/5} = 1, -2 \leq x \leq -1\}$ ;  $[(-2, 2), (-2, 1)]$ ;  $\{y^{1/5} + (2+x)^{1/5} = 1, -2 \leq x \leq -1\}$  and  $[(-1, 0), (1, 0)]$ . Note that  $H_1 \subset H_2 \subset [-2, 2] \times [0, 3]$ ,  $\partial H_1 \cap \mathbb{R} = \partial H_2 \cap \mathbb{R} = [-1, 1]$  and that  $\partial H_1, \partial H_2 \in C^3$ .

Let  $\Gamma$  be an orientation preserving arc-length parameterization of  $\gamma = \partial D$  such that for  $s_0 \in (0, \text{length}(\gamma))$  there holds:  $\Gamma(s_0) = f(t)$ . Let  $D^* = \overline{\Gamma'(s_0)D}$ ,  $b = \overline{\Gamma'(s_0)f(t)}$  and  $\Gamma^* = \overline{\Gamma'(s_0)\Gamma}$ . Then there exists  $r > 0$  such that  $(b, b+ir] \subset D^*$ . Since  $\gamma^* = \partial D^* \in C^{2,\alpha}$ , it follows that, there exist  $x_0 > 0$ ,  $\varepsilon > 0$ ,  $y_0 \in (0, r/3)$ , the  $C^{2,\alpha}$  function  $h : [-2x_0, 2x_0] \rightarrow \mathbb{R}$ ,  $h(0) = 0$ , and the domain  $D_2^* \subset D^*$  such that:

- (1)  $\Gamma^*([s_0 - \varepsilon, s_0 + \varepsilon]) = \{b + (x, h(x)) : x \in [-2x_0, 2x_0]\}$ ,
- (2)  $D_2^* = \{b + (x, h(x) + y) : x \in [-2x_0, 2x_0], y \in (0, 3y_0]\}$ .

Let  $\Upsilon : [-2, 2] \times [0, 3] \rightarrow D_2^*$  be the mapping defined by:

$$\Upsilon(x, y) = b + (xx_0, h(xx_0) + yy_0).$$

Then  $\Upsilon$  is a  $C^{2,\alpha}$  diffeomorphism.

Take  $D_i = \Gamma'(s_0) \cdot \Upsilon(H_i)$ ,  $i = 1, 2$ . Obviously  $D_1 \subset D_2 \subset D$  and  $D_1$  and  $D_2$  have  $C^{2,\alpha}$  boundary. Observe that  $f(t) = \Gamma'(s_0)\overline{\Gamma'(s_0)}f(t) = \Gamma'(s_0)\Upsilon(0) \in \Gamma'(s_0)\Upsilon([-1, 1]) = \partial D_1 \cap \partial D_2$ .

**Step 2 (Application of Heinz-Bernstein theorem).** Let  $\phi$  be a conformal mapping of  $D_2$  onto  $H$  such that  $\phi^{-1}(\infty) \in \partial D_2 \setminus \partial D_1$ . Let  $\Omega_1 = \phi(D_1)$ . Then there exist real numbers  $a, b, c, d$  such that  $a < c < d < b$ ,  $[a, b] = \partial\Omega_1 \cap \mathbb{R}$  and  $l = \phi^{-1}(\partial\Omega_1 \setminus [c, d]) \subset D$ . Let  $U_1 = f^{-1}(D_1)$  and  $\eta$  be a conformal mapping between the unit disc and the domain  $U_1$ . Then the mapping  $\hat{f} = \phi \circ f \circ \eta$  is a  $C^2$  diffeomorphism of the unit disc onto the domain  $\Omega_1$  such that:

- (a)  $\hat{f}$  is continuous on the boundary  $\mathbb{T} = \partial\mathbb{U}$  (it is q.c.) and
- (b)  $\hat{f}$  is  $C^2$  on the set  $T_1 = \hat{f}^{-1}(\partial\Omega_1 \setminus (c, d))$ .

Let  $s := \text{Im } \hat{f}$ . First, note that (a) implies that  $s$  is continuous on  $\mathbb{T} = \partial\mathbb{U}$ . On other hand, as  $\hat{f} \in C^2$ ,  $s$  satisfies the condition:

- (1)  $s \in C^2(\mathbb{U})$ .

From (b) we obtain that  $s$  is  $C^2$  on the set  $T_1 = \hat{f}^{-1}(\partial\Omega \setminus (c, d))$ . Furthermore,  $s = 0$  on  $T_2 = \hat{f}^{-1}(a, b)$ ; and therefore  $s$  is  $C^2$  on  $T_2 = \hat{f}^{-1}(a, b)$ . Hence:

- (2)  $s$  is  $C^2$  on  $\mathbb{T} = T_1 \cup T_2$ . In other words, the function  $s_b : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $s_b(\theta) = s(e^{i\theta})$  is  $C^2$  in  $\mathbb{R}$ .

In order to apply the interior estimate, we have to prove that

- (3)  $|\Delta s(z)| \leq M_0 |\nabla s(z)|^2 + N_0$ ,  $z \in \mathbb{U}$ , where  $M_0$  and  $N_0$  are constants.

To continue we need the following lemma:

**Lemma 3.3.** *If  $f = u + iv$  is a q.c. mapping satisfying Poisson differential inequality, then  $u$  and  $v$  satisfy the Poisson differential inequality.*

*Proof.* Let

$$A := |\nabla u|^2 = 2(|u_z|^2 + |u_{\bar{z}}|^2) = \frac{1}{2}(|f_z + \overline{f_{\bar{z}}}|^2 + |f_{\bar{z}} + \overline{f_z}|^2)$$

and

$$B := |\nabla v|^2 = 2(|v_z|^2 + |v_{\bar{z}}|^2) = \frac{1}{2}(|f_z - \bar{f}_{\bar{z}}|^2 + |f_{\bar{z}} - \bar{f}_z|^2).$$

Then

$$\frac{A}{B} = \frac{|1 + \mu|^2}{|1 - \mu|^2}$$

where  $\mu = \bar{f}_{\bar{z}}/f_z$ . Since  $|\mu| \leq k$

$$(3.2) \quad \frac{(1-k)^2}{(1+k)^2} \leq \frac{A}{B} \leq \frac{(1+k)^2}{(1-k)^2}.$$

As

$$|\Delta f| = |\Delta u + i\Delta v| \leq M|\nabla f|^2 + N = M(|\nabla u|^2 + |\nabla v|^2) + N,$$

the relation (3.2) yields

$$|\Delta u| \leq M \frac{(1+k)^2}{(1-k)^2} |\nabla u|^2 + N$$

and

$$|\Delta v| \leq M \frac{(1+k)^2}{(1-k)^2} |\nabla v|^2 + N.$$

□

Since  $\hat{f} = \phi \circ f \circ \eta$ , we obtain

$$(3.3) \quad \partial \hat{f} = \phi' \partial f \eta', \quad \bar{\partial} \hat{f} = \phi' \bar{\partial} f \eta'$$

and

$$(3.4) \quad \partial \bar{\partial} \hat{f} = \frac{1}{4} \Delta \hat{f} = \frac{1}{4} \Delta(\phi \circ f) \cdot |\eta'|^2 = (\phi'' \partial f \cdot \bar{\partial} f + \phi' \partial \bar{\partial} f) |\eta'|^2.$$

Now combining (3.1), (3.3) and (3.4) we obtain

$$\begin{aligned} |\Delta \hat{f}| &\leq 4 \frac{|\phi''|}{|\phi'|^2} |\partial \hat{f}| |\bar{\partial} \hat{f}| + |\phi'| |\Delta f| |\eta'|^2 \\ &\leq 4 \frac{|\phi''|}{|\phi'|^2} |\partial \hat{f}| |\bar{\partial} \hat{f}| + |\phi'| (M |\nabla f|^2 + N) |\eta'|^2 \\ &\leq \frac{|\phi''|}{|\phi'|^2} |\nabla \hat{f}|^2 + M |\nabla \hat{f}|^2 \cdot \frac{1}{|\phi'|} + N |\phi'| |\eta'|^2 \\ &= \left( \frac{|\phi''|}{|\phi'|^2} + \frac{M}{|\phi'|} \right) |\nabla \hat{f}|^2 + N |\phi'| |\eta'|^2. \end{aligned}$$

As  $\hat{f}$  is a  $k$ -q.c. mapping using Lemma 3.3 we have

$$(3.5) \quad |\Delta s| \leq \frac{(1+k)^2}{(1-k)^2} \left( \frac{|\phi''|}{|\phi'|^2} + \frac{M}{|\phi'|} \right) \cdot |\nabla s|^2 + N |\phi'| |\eta'|^2.$$

Proposition 2.1 implies that the function  $|\eta'|$  is bounded from above by a constant  $C_1$ , the function  $|\phi'|$  is bounded from below and above by positive constants  $C_2$  and  $C_3$  respectively and the function  $|\phi''|$  is bounded from above by a constant  $C_4$ . Hence

$$|\Delta s| \leq M_0 |\nabla s|^2 + N_0,$$

where

$$M_0 = \frac{(1+k)^2}{(1-k)^2} \left( \frac{C_4}{C_3^2} + \frac{M}{C_3} \right) \text{ and } N_0 = C_2 C_1^2 N.$$

Proposition 3.2 implies that, the function  $|\nabla s|$  is bounded by a constant  $b_t$ . Since  $\hat{f}$  is a  $k$ -q.c. mapping, we have

$$(1-k)|\partial\hat{f}| \leq |\partial\hat{f} - \overline{\partial\hat{f}}| \leq 2|s_z| \leq \sqrt{2}b_t.$$

Finally,

$$|\partial\hat{f}| + |\bar{\partial}\hat{f}| \leq \sqrt{2} \frac{1+k}{1-k} b_t.$$

Since the mapping  $\eta$  is conformal and maps the circle arc  $T = (\phi \circ f \circ \eta)^{-1}(a, b)$  onto the circle arc  $(\phi \circ f)^{-1}(a, b)$ , it follows that it can be conformally extended across the arc  $T' = (\phi \circ f \circ \eta)^{-1}[c, d]$ . Hence, there exists a constant  $A$  such that  $|\eta'(z)| \geq 2A$  on  $T'$ . It follows that there exists  $r \in (0, 1)$  such that  $|\eta'(z)| \geq A$  in  $\mathcal{T} = \{\rho z : z \in T', r \leq \rho \leq 1\}$ . It follows from the Proposition 2.1, that the conformal mapping  $\phi$  and its inverse have the  $C^1$  extension to the boundary. Therefore there exists a positive constant  $B$  such that  $|\phi'(z)| \geq B$  on some neighborhood of  $\phi^{-1}[c, d]$  with respect to  $D$ . Thus, the mapping  $f = \phi^{-1} \circ \hat{f} \circ \eta^{-1}$  has bounded derivative in some neighborhood of the set  $\eta(T')$ , on which it is bounded by the constant

$$C = \sqrt{2} \frac{1+k}{1-k} \frac{b_t}{AB}.$$

Then

$$|\partial f(z)| + |\bar{\partial} f(z)| \leq C_0 \text{ for all } z \in \mathbb{U} \text{ near the arc } T = \eta(T').$$

□

**Theorem 3.4.** *Let  $f$  be a quasiconformal  $C^2$  diffeomorphism from the plane domain  $\Omega$  with  $C^{1,\alpha}$  compact boundary onto the plane domain  $G$  with  $C^{2,\alpha}$  compact boundary. If there exist constants  $M$  and  $N$  such that*

$$(3.6) \quad |\Delta f| \leq M|\nabla f|^2 + N, \quad z \in \Omega,$$

*then  $f$  has bounded partial derivatives in  $\Omega$ . In particular it is a Lipschitz mapping in  $\Omega$ .*

*Proof.* According to the Theorem 3.1 for every  $t \in \partial\Omega$  there exists  $r_t > 0$  such that  $f$  has bounded partial derivatives in  $\Omega \cap D(t, r_t)$ . Since  $\partial\Omega$  is a compact set it follows that there exists  $t_1, \dots, t_m$  such that  $\partial\Omega \subset \bigcup_{i=1}^m D(t_i, r_{t_i})$ . It follows that  $f$  has bounded partial derivatives in  $\Omega \cap \bigcup_{i=1}^m D(t_i, r_{t_i})$ . Since  $f$  is diffeomorphism in  $\Omega$  we obtain that  $f$  has bounded derivatives in the compact set  $\Omega \setminus \bigcup_{i=1}^m D(t_i, r_{t_i})$ . The conclusion of the theorem now easily follows. □

**Corollary 3.5.** *Let  $\Omega$  be a plane domain with  $C^{1,\alpha}$  compact boundary and  $G$  be a plane domain with  $C^{2,\alpha}$  compact boundary. If  $w = f(z) : \Omega \mapsto G$  is quasiconformal solution of the equation*

$$(3.7) \quad \begin{aligned} &\alpha w_{xx} + 2\beta w_{xy} + \gamma w_{yy} + a_1(z)w_x^2 + b_1(z)w_x w_y + c_1(z)w_y^2 \\ &+ a(z)w_x + b(z)w_y + c(z)w + d(z) = 0, \end{aligned}$$

*such that  $\alpha, \beta, \gamma \in \mathbb{R}$ ,  $\alpha > 0$ ,  $\alpha\gamma - \beta^2 > 0$ ,  $a, b, c, d, a_1, b_1, c_1 \in C(\overline{\Omega})$ , then  $f$  is Lipschitz.*

*Proof.* Since the PDE (3.7) is elliptic, we can choose coordinates  $x = \alpha_1 u + \beta_1 v$ ,  $y = \beta_1 u + \gamma_1 v$  such that (3.7) becomes

$$(3.8) \quad \begin{aligned} & w_{uu} + w_{vv} + a'_1(u, v)w_u^2 + b'_1(u, v)w_u w_v + c'_1(u, v)w_v^2 \\ & + a'(u, v)w_u + b'(u, v)w_v + c'(u, v)w + d'(u, v) = 0, \quad (u, v) \in \Omega'. \end{aligned}$$

For  $e \in C(\overline{\Omega'})$  let  $|e| = \max\{|e(u, v)| : (u, v) \in \overline{\Omega'}\}$ . Using (3.8) and the inequality  $2|t| \leq |t|^2 + 1$  we obtain

$$\begin{aligned} |\Delta w| &\leq \left( \frac{|a'|}{2} + \frac{|b'|}{2} \right) (|\nabla w|^2 + 1) + \left( \max\{|a'_1|, |c'_1|\} + \frac{|b'_1|}{2} \right) |\nabla w|^2 + |c'| |w| + |d'| \\ &= M |\nabla w|^2 + N, \end{aligned}$$

where

$$M = (|a'| + |b'|)/2 + \max\{|a'_1|, |c'_1|\} + \frac{|b'_1|}{2}$$

and

$$N = \frac{|a'| + |b'|}{2} + |c'| |w| + |d'|.$$

The conclusion now follows from Theorem 3.4.  $\square$

By  $d = d_k$  we denote Euclidean distance in Euclidean space  $\mathbb{R}^k$ .

**Theorem 3.6.** *We call a  $C^{2,\alpha}$  surface  $S$  disk-like surface if it is homeomorphic to the unit disk, and if its boundary is a  $C^{2,\alpha}$  curve. If  $f$  is a quasiconformal harmonic mapping between two  $C^{2,\alpha}$  disk-like surfaces  $S_1$  and  $S_2$ , then it is a Lipschitz mapping i.e. there exists a constant  $C$  such that*

$$d(f(x), f(y)) \leq Cd(x, y), \text{ for all } x, y \in S_1.$$

*Proof.* Let  $f$  be a harmonic q.c. mapping between disk-like surfaces  $S_1$  and  $S_2$ . Let  $X : \mathbb{U} \mapsto S_1$  and  $Y : \mathbb{U} \mapsto S_2$  be conformal mappings. Let us consider the mapping  $g = Y^{-1} \circ f \circ X$  of the unit disk onto itself. Since  $f(X(z)) = Y(g(z))$ , it follows that

$$|f \circ X_x|^2 + |f \circ X_y|^2 = |Y_u|^2 (|g_x|^2 + |g_y|^2).$$

Hence

$$(3.9) \quad E[f \circ X] = E_Y[g] = \iint_{\Omega} |Y_u|^2 (|g_x|^2 + |g_y|^2) dx dy.$$

If we denote  $\rho(w) = |Y_u(w)|^2$ , then the stationary points of the energy integral  $E_Y[g]$  satisfies the Euler-Lagrange equation

$$(3.10) \quad g_{z\bar{z}} + (\log \rho)_w \circ g g_z g_{\bar{z}} = 0.$$

Consequently  $f \circ X$  is harmonic if and only if  $g$  is  $\rho$ -harmonic i.e. the mapping satisfying the relation (3.10). According to the Corollary 2.3, the mapping  $g$  satisfies the conditions of Theorem 3.4. Namely as  $|(\log \rho)_w| \leq M$  and  $|g_z g_{\bar{z}}| \leq 1/2(|g_z|^2 + |g_{\bar{z}}|^2)$  we can simply take  $M = M'/2$ , and  $N = 0$ . Theorem 3.4 yields that  $g$  is Lipschitz. By Theorem 2.2 it follows that  $X$  and  $Y$  are bi-Lipschitz mapping.  $f$  is Lipschitz as a composition of Lipschitz mappings.  $\square$

Using Theorem 3.6 we obtain the theorem:

**Theorem 3.7.** *If  $f$  is a quasiconformal harmonic mapping between  $C^{2,\alpha}$  surfaces  $S_1$  and  $S_2$ , with  $C^{2,\alpha}$  compact boundary then it is a Lipschitz mapping i.e. there exists a constant  $C$  such that  $d(f(x), f(y)) \leq Cd(x, y)$ , for all  $x, y \in S_1$ .*

**3.1. A question.** Recently in [6] is proved that, every quasiconformal harmonic mapping between two Jordan domains  $\Omega_j$ ,  $j = 1, 2$ , with  $C^{j,\alpha}$ ,  $j = 1, 2$  boundary is bi-Lipschitz, and this can be extended directly to the plane domains with  $C^{j,\alpha}$ ,  $j = 1, 2$  boundary. The question arises, whether the previous statement can be extended to  $C^{j,\alpha}$  surfaces?

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