INNER ESTIMATE AND QUASICONFORMAL HARMONIC MAPS BETWEEN SMOOTH DOMAINS

DAVID KALAJ AND MIODRAG MATELJEVIĆ

Abstract. We prove a version of "inner estimate" for quasi-conformal diffeomorphisms, which is satisfying certain estimate concerning their laplacian. As an application of this estimate, we show that quasiconformal harmonic mappings between smooth domains (with respect to the approximately analytic metric), have bounded partial derivatives; in particular, these mappings are Lipschitz.

Our discussion includes harmonic mappings with respect to (a) spherical and euclidean metrics (which are approximately analytic) as well as (b) the metric induced by the holomorphic quadratic differential.

1. Introduction and statement of the main result

1.1. **Basic facts and notations.** $\mathbb{U}$ and $\mathbb{H}$ denote, respectively the unit disc and the upper half plane. By Ω, Ω' and D we denote the simply connected domains. Suppose that γ is a rectifiable curve in the complex plane or in Riemann surface R. Denote by l the length of γ and let $\Gamma : [0,l] \mapsto \gamma$ be the natural parameterization of γ, i.e. the parameterization satisfying the condition:

$$|\dot{\Gamma}(s)| = 1 \text{ for all } s \in [0,l].$$

We will say that γ is of class $C^{n,\mu}$, for $n \in \mathbb{N}$, $0 < \mu \leq 1$, if Γ is of class C^n and

$$\sup_{t,s} \frac{|\Gamma^{(n)}(t) - \Gamma^{(n)}(s)|}{|t-s|^{\mu}} < \infty.$$

Jordan domains in $\mathbb{C}$ bounded by $C^{n,\mu}$ Jordan curves, we will call $C^{n,\mu}$ domains or smooth ones.

Let $\rho(w)|dw|^2$ be an arbitrary conformal C^1-metric defined on D. If $f : \Omega \to D$ is a C^2 mapping between Jordan domains Ω and D, the energy integral of f is defined by the formula:

$$E[f,\rho] = \int_{\Omega} \rho \circ f(|f_z|^2 + |f_{\bar{z}}|^2)dx\,dy. \tag{1.1}$$

2000 *Mathematics Subject Classification.* Primary 30C55; Secondary 30C62.

Key words and phrases. Quasiconformal harmonic maps, Jordan domains, Riemann surface, Spherical metric, Poincaré metric, Lipschitz condition.

The stationary points of the energy integral satisfy the Euler Lagrange equation

$$f_{z\overline{z}} + (\log\rho)_w \circ f f_z\, f_{\bar z} = 0, \tag{1.2}$$

and a C^2 solution of this equation is a called *harmonic mapping* (more precisely a *ρ-harmonic mapping*).

It is known that f is a harmonic mapping if and only if the mapping

$$\Psi = \rho \circ f f_z \overline{f}_{\bar z} \tag{1.3}$$

is analytic; and we say that Ψ is the *Hopf differential* of f and we write Ψ =Hopf(f).

If φ is a holomorphic mapping and different from 0 on D and $\rho = |\varphi|$ on D, we call ρ a $\varphi-$ metric.
The corresponding harmonic mapping we will call *$\varphi-$harmonic.*
Notice that for $\rho = 1$, ρ-harmonic mapping is an Euclidean harmonic function.

Since we consider diffeomorphisms or C^2-mappings, we can use the following definition of quasiconformal mappings.
Let $0 \le k < 1$ and $K = \frac{1+k}{1-k}$. An orientation preserving diffeomorphism $f : \Omega \mapsto D$ between two domains $\Omega, D \subset \mathbb{C}$ is called K *-quasiconformal mapping* or shortly a q.c. mapping if it satisfies the condition:

$$|f_{\overline{z}}(z)| \le k|f_z(z)| \text{ for each } z \in \Omega. \tag{1.4}$$

Occasionally, in this setting, it is also convenient to say that f is a k *-quasiconformal mapping.*

In this paper we will mainly consider harmonic quasiconformal mappings between smooth domains.

1.2. **Background.** It seems that Martio ([17]) was the first one who considered the problem of characterization of harmonic quasiconformal mappings w.r. to the Euclid metric for the unit disc. Now there is a number of results related to this topic.

Theorem P. *If w is a harmonic diffeomorphism of the unit disc onto itself, then the following conditions are equivalent: w is q.c.; w is bi-Lipschitz; the boundary function is bi-Lipschitz and the Hilbert transformation of its derivative is in L^∞.* See [22].

Theorem KP. *An orientation-preserving homeomorphism ψ of the real axis can be extended to a q.c. harmonic homeomorphism of the upper half-plane if and only if ψ is bi-Lipschitz and the Hilbert transformation of the derivative ψ' is bounded.* See [13] and [10].

Theorem K. *If Ω and Ω' are Jordan domains with $C^{1,\mu}$ boundary* ($0 < \mu \le 1$)*, then every quasiconformal harmonic function from Ω onto Ω' is Lipschitz. If Ω' is convex, then w is bi-Lipschitz. Moreover if $w : \Omega \mapsto \Omega'$ is a harmonic diffeomorphism, where Ω is the unit disc and Ω' is a convex domain with $C^{1,\mu}$ boundary then the following conditions are equivalent: w*

is quasiconformal; w is bi-Lipschitz; the boundary function is bi-Lipschitz and the Hilbert transformation of its derivative is in L^∞; and therefore a harmonic diffeomorphism in this setting is a quasi-isometry w.r. to the corresponding Poincaré distance. See [14].

Concerning quasi-isometry it seems that the second author in joint work with M. Knežević obtained the right constant (See [16]). Namely, they proved that:

Theorem MK1. *If f is $K-$q.c. harmonic diffeomorphism from the upper half plane H onto itself and $f(\infty)=\infty$, then*

$$|z_1-z_2|/K \le |f(z_1)-f(z_2)| \le K|z_2-z_1| \quad \text{where } z_1,z_2\in H$$

and f is $(1/K,K)$ quasi-isometry w.r. to Poincaré distance.

Theorem MK2. *If f is $K-$q.c. harmonic diffeomorphism from the unit disc U onto itself, then f is $(1/K,K)$ quasi-isometry w.r. to Poincaré distance:*

$$d_h(z_1,z_2)/K \le d_h(f(z_1),f(z_2)) \le K d_h(z_1,z_2),$$

where d_h is hyperbolic distance in the unit disc.
Concerning the hyperbolic q.c. harmonic mapping we present here two results:
Theorem W1. *Every harmonic quasi-conformal mapping from the unit disc onto itself is a quasi-isometry of Poincaré disc.* See [28].

Theorem W2. *A harmonic diffeomorphism of the hyperbolic plane $\mathbb{H}^2$ is quasiconformal if and only if its Hopf differential is uniformly bounded with respect to the Poincaré metric.* See [28].

For the other results in this area see [19],[16],[7], [25], [29], [26], [21] and [2].

1.3. **New results.** The following proposition has an important role in the proofs concerning results obtained in [10], [13] and [16].

Proposition 1.1. *If f is an Euclidean harmonic $1-1$ mapping of the upper half-plane $\mathbb{H}$ onto itself, normalized by $f(\infty)=\infty$ and $v=\mathrm{Im} f$, then $v(z)=cy$, where c is a positive constant. In particular, v has bounded partial derivatives on $\mathbb{H}$.*

Suppose that f is a harmonic Euclidean mapping of the unit disc onto smooth domain D and ψ is conformal mapping of D onto $\mathbb{H}$, then the composition $\psi\circ f$ is very rarely Euclidean harmonic, so we can not apply the Proposition 1.1. However, the composition satisfies a simple equation (see (4.3), Section 4) and it is harmonic with respect to the other metric ρ defined on $\mathbb{H}$ by $\rho(\zeta)|d\zeta|=|dw|$, where $\zeta=\psi(w)$. Having this in mind, our idea is to apply the following essential generalization of Proposition 1.1:

Proposition 1.2 (Inner Estimate)**. (Heinz-Bernstein, see** [9]**).** *Let $s:\overline{\mathbb{U}}\mapsto\mathbb{R}$ be a continuous function from the closed unit disc $\overline{\mathbb{U}}$ into the real line satisfying the conditions:*

(1) *s is C^2 on $\mathbb{U}$,*
(2) *$s_b(\theta) = s(e^{i\theta})$ is C^2 and*
(3) *$|\Delta s| \leq c_0|\bigtriangledown s|^2$ on $\mathbb{U}$ for some constant c_0.*

Then the function $|\bigtriangledown s| = |\mathrm{grad}\, s|$ is bounded on $\mathbb{U}$.

We refer to this result as the *inner estimate.* Applying this estimate and Kellogg - Warshawski results we prove the main result of this paper.

Theorem 1.3 (Main Result)**.** *Let f be a quasiconformal C^2 diffeomorphism from the $C^{1,\alpha}$ Jordan domain Ω onto the $C^{2,\alpha}$ Jordan domain D. If there exists a constant M such that*

$$|\Delta f| \leq M|f_z \cdot f_{\bar{z}}|\,, \quad z \in \Omega, \tag{1.5}$$

then f has bounded partial derivatives. In particular, it is a Lipschitz mapping.

Notify that equation (2.3) (see below) shows how to transform condition (1.5) if we consider compositions of the mapping f by conformal mappings. In particular, Theorem 1.3 holds if h is quasiconformal ρ-harmonic and the metric ρ is *approximately analytic,* i.e. $|\bar{\partial}\rho| \leq M|\rho|$ on Ω, (see Theorems 3.1-3.2, 3.4, 4.4 below).
Notice that

(a) Theorem 3.1 can be considered as a special case of Theorem 3.4 and 4.4 and
(b) Euclidean and spherical metrics are approximately analytic, so our results can be considered as extensions of the corresponding ones proved in [17], [22], [13], [10] and [14] (mentioned in subsection 1.2).

The paper is organized as follows. The main result is proved in Section 2 and its applications are given in Section 3.
In Section 4, we show that the composition of a conformal mapping ψ and a $\varphi-$ harmonic mapping satisfies a certain equation (see Theorem 4.1); and in particular, if ψ is a natural parameter, we obtain a representation of $\varphi-$harmonic mappings by means of Euclidean harmonics. As applications, we produce some examples of $\varphi-$harmonic mappings and prove that Theorem 3.1 holds for more general domains.

2. The proof of main result

We need the following propositions:

Proposition 2.1 (Kellogg)**.** *See for example* [8]*. If a domain $D = Int(\Gamma)$ is $C^{1,\alpha}$ and ω is a conformal mapping of $\mathbb{U}$ onto D, then ω' and $\ln \omega'$ are in Lip_α. In particular, $|\omega'|$ is bounded from above and below on U.*

Proposition 2.2 (Kellogg and Warschawski)**.** *See* [23, Theorem 3.6]*. If a domain $D = Int(\Gamma)$ is $C^{2,\alpha}$ and ω is a conformal mapping of $\mathbb{U}$ onto D, then $|\omega''|$ has a continuous extension to the boundary. In particular it is bounded from above on $\mathbb{U}$.*

Proof of Theorem 1.3. Let g be a conformal mapping of the unit disc onto Ω. Let $\tilde{f} = f \circ g$. Since $\Delta \tilde{f} = |g'|^2 \Delta f$ and $\tilde{f}_z \cdot \tilde{f}_{\bar{z}} = |g'|^2 f_z \cdot f_{\bar{z}}$, we obtain that $\tilde{f}$ satisfies the inequality (1.5). We will prove the theorem for $\tilde{f}$ and then apply Kellogg's theorem. To simplify notations, we will use f instead of $\tilde{f}$. Let $t \in \partial U = T$ be an arbitrary fixed point. In order to continue the proof (to apply inner estimate), we use the following procedure which we call local construction:

Step 1 (**Local Construction**): There are two Jordan domains D_1 and D_2 in D with $C^{2,\alpha}$ boundary such that

(i) $D_1 \subset D_2 \subset D$,
(ii) $\partial D \cap \partial D_2 = l$ is connected arc containing the point $w = f(t)$ in its interior,
(iii) $\partial D_1 \cap \partial D_2$ is a connected arc containing l and $\emptyset \neq \overline{\partial D_2 \setminus \partial D_1} \subset D$.

It is convenient for the reader to make a picture to visualize this construction. We can obtain an example of the sets D_1 and D_2 by following procedure. Let ψ be a conformal mapping of D onto the upper half-plane such that $\psi(f(t)) \neq \infty$. Choose now two C^3 subdomains H_1 and H_2 of H such that $H_1 \subset H_2$ and $\partial H_1 \cap \mathbb{R} = [c_0, d_0] \subset \partial H_2 \cap \mathbb{R} = [a_0, b_0]$, $a_0 < c_0 < \psi(f(t)) < d_0 < b_0$. Then we take $D_i = \psi^{-1}(H_i)$, $i = 1, 2$.

Step 2 (**Application of the Inner Estimate**). Let ϕ be a conformal mapping of D_2 onto H such that $\phi^{-1}(\infty) \in \partial D_2 \setminus \partial D_1$. Let $\Omega^* = \phi(D_1)$. Then there exist real numbers a, b, c, d such that $a < c < d < b$, $[a, b] = \partial\Omega \cap \mathbb{R}$ and $\ell = \phi^{-1}(\partial\Omega^* \setminus [c, d]) \subset D$. Let $U_1 = f^{-1}(D_1)$ and η be a conformal mapping between the unit disc and the domain U_1. Then the mapping $\hat{f} = \phi \circ f \circ \eta$ is a C^2 diffeomorfism of the unit disc onto the domain Ω^* such that:

(a) $\hat{f}$ is continuous on the boundary $T = \partial U$ (it is q.c.) and
(b) $\hat{f}$ is C^2 on the set $T_1 = \hat{f}^{-1}(\partial\Omega \setminus (c, d))$.

Let $s := \hat{v} = \operatorname{Im} \hat{f}$. Note first that, because of (a), s is continuous on $T = \partial U$. On other hand, since $\hat{f} \in C^2$, s satisfies the condition:

(1) $s \in C^2(U)$.

From (b) we obtain that s is C^2 on the set $T_1 = \hat{f}^{-1}(\partial\Omega \setminus (c, d))$. Further s satisfies the condition: $s = 0$ on $T_2 = \hat{f}^{-1}(a, b)$; and therefore s is C^2 on $T_2 = \hat{f}^{-1}(a, b)$. Hence we have:

(2) s is C^2 on $T = T_1 \cup T_2$. In other words, the function $s_b : \mathbb{R} \mapsto \mathbb{R}$ defined by $s_b(\theta) = s(e^{i\theta})$ is C^2 in $\mathbb{R}$.

In order to apply the inner estimate, we have to prove that the function s satisfies the condition

(3) $|\Delta s(z)| \leq c_0 | \bigtriangledown s(z)|^2$, $z \in U$, where c_0 is a constant.

We start with the equalities:

$$s_z = \frac{\hat{f}_z - \overline{\hat{f}_{\bar{z}}}}{2i} \text{ and } s_{\bar{z}} = \frac{\hat{f}_{\bar{z}} - \overline{\hat{f}_z}}{2i}, \tag{2.1}$$

which imply the equality

$$\overline{s_z} = s_{\bar z}$$

and therefore

$$|s_z|^2 = |s_{\bar z}|^2 = \frac{|\bigtriangledown s|^2}{2}. \tag{2.2}$$

Since $\hat f = \phi \circ f \circ \eta$, we obtain the equality (see also Theorem 4.1, Section 4 below)

$$\frac{\hat f_{z\bar z}}{\hat f_z \cdot \hat f_{\bar z}} = \left(\frac{\phi''}{\phi'^2} + \frac{1}{\phi'}\frac{\partial\bar\partial f}{\partial f \cdot \bar\partial f}\right). \tag{2.3}$$

Observe that $\hat f$ is a k- q.c. mapping. Hence

$$|\Delta s| = |\mathrm{Im}\,\Delta \hat f| \le |\hat f_z| \cdot |\hat f_{\bar z}| \cdot \left|\frac{\phi''}{\phi'^2} + \frac{1}{\phi'}\frac{\partial\bar\partial f}{\partial f \cdot \bar\partial f}\right| \le k|\hat f_z|^2 \left|\frac{\phi''}{\phi'^2} + \frac{1}{\phi'}\frac{\partial\bar\partial f}{\partial f \cdot \bar\partial f}\right|.$$

Using (2.1) and (2.2) respectively, we obtain that

$$(1-k)|\hat f_z| \le 2|s_z|,$$

and

$$|\Delta s| \le \frac{2k}{(1-k)^2}\left|\frac{\phi''}{\phi'^2} + \frac{1}{\phi'}\frac{\partial\bar\partial f}{\partial f \cdot \bar\partial f}\right| \cdot |\bigtriangledown s|^2. \tag{2.4}$$

By applying Proposition 2.1 and the Proposition 2.2, it follows that the function $|\phi'|$ is bounded from below by a positive constant C_1 and the function is $|\phi''|$ bounded from above by a constant C_2. Now using (1.5), we obtain the inequality:

$$\left|\frac{\phi''}{\phi'^2} + \frac{1}{\phi'}\frac{\partial\bar\partial f}{\partial f \cdot \bar\partial f}\right| \le \frac{4C_2 + MC_1}{4C_1^2}. \tag{2.5}$$

Combing (2.4) and (2.5), we conclude that the function s satisfies the inequality

$$|\Delta s| \le c_0 |\bigtriangledown s|^2,$$

where

$$c_0 = \frac{2k}{(1-k)^2} \cdot \frac{4C_2 + MC_1}{4C_1^2}.$$

Now from Proposition 1.2 we derive that the function $|\bigtriangledown s|$ is bounded by a constant b_t. Since $\hat f$ is a $k-$q.c. mapping, we have

$$(1-k)|\hat f_z| \le |\hat f_z - \overline{\hat f_{\bar z}}| \le 2|s_z| \le \sqrt{2} b_t.$$

Finally, we obtain the inequality

$$|\hat f_z| + |\hat f_{\bar z}| \le \sqrt{2}\frac{1+k}{1-k} b_t.$$

Put $T^t_{c,d} = (f\circ\eta)^{-1}(c,d)$. Observe that c and d depend on the fixed point t. Since $t \in T^t_{c,d}$ we obtain $T = \bigcup_{t\in T} T^t_{c,d}$, and therefore there exists a finite set $\{t_1,\dots,t_n\}$ such that $T = \bigcup_{i=1}^n T^{t_i}_{c,d}$.

Since the mapping $\eta_i = \eta$ is conformal and maps the circle arc $T_i = (\phi\circ f\circ\eta)^{-1}(a,b)$ onto the circle arc $(\phi\circ f)^{-1}(a,b)$, it follows that it can be conformally extended across the arc $T'_i = (\phi\circ f\circ\eta)^{-1}[c,d]$, and hence there exists a constant A_i such that $|\eta'(z)| \geq 2A_i$ on T'_i. It follows that there exists $r_i \in (0,1)$ such that $|\eta'(z)| \geq A_i$ in $\mathcal{T}_i = \{\rho z : z \in T'_i, r_i \leq \rho \leq 1\}$. Also, applying Proposition 2.1, we infer that the conformal mapping $\phi_i = \phi$ and its inverse have the C^1 extension to the boundary. It follows that there exists a positive constant B_i such that $|\phi'(z)| \geq B_i$ on some neighborhood of $\phi^{-1}[c,d]$ with respect to D. We obtain that the mapping $f = \phi^{-1}\circ\hat{f}\circ\eta^{-1}$ has bounded derivative in some neighborhood of the set $T^{t_i}_{c,d}$, on which it is bounded by the constant

$$C_i = \sqrt{2}\frac{1+k}{1-k}\frac{b_{t_i}}{A_iB_i}.$$

Set $C_0 = \max\{C_1,\dots,C_n\}$. Then

$$|f_z(z)| + |f_{\bar z}(z)| \leq C_0 \text{ for all } z \in U \text{ near to } T = \partial U.$$

Since f is diffeomorphism in U, we obtain the desired conclusion. $\square$

3. Applications

Let D be a domain in $\mathbb{C}$ and ρ be a conformal metric in D. The Gaussian curvature on the domain is given by

$$K_D = -\frac{1}{2}\frac{\Delta\log\rho}{\rho}\,.$$

If, in particular, the domain D is a simply connected in $\mathbb{C}$ and the Gaussian curvature $K_D = 0$ on D, then $\Delta\log\rho = 0$ and therefore $\rho = |e^{\omega}|$, where ω is a holomorphic function on D.

Thus the metric ρ is induced by non-vanishing holomorphic function $\varphi(z) = e^{\omega(z)}$ defined on the domain D; in this setting we call ρ a $\varphi-$ metric. The corresponding harmonic mapping we will call *$\varphi-$harmonic.*
Roughly speaking, *$\varphi-$harmonic* maps arise if the curvature of target is 0.

Since $\rho^2 = \varphi\overline{\varphi}$, a short computation yields $2\rho\rho_w = \varphi'\overline{\varphi}$ and therefore $2(\log\rho)_w = (\log\varphi)'$. Hence, by (1.2) we obtain: if f is $\varphi-$ harmonic, then

$$f_{z\bar z} + \frac{\varphi'}{2\varphi}\circ f f_z\, f_{\bar z} = 0. \tag{3.1}$$

As a direct application of Theorem 1.3 (the main result), using the equality (3.1), we obtain the following theorem:

Theorem 3.1. *Let f be a $\varphi-$harmonic mapping of the unit disc U onto a $C^{2,\alpha}$ Jordan domain D. If $M = ||(\log\varphi)'||_\infty < \infty$ and f is quasiconformal,* *then f has bounded partial derivatives and in particular, it is a Lipschitz mapping.*

Proof. It is enough to notice that the hypothesis $M = ||(\log\varphi)'||_\infty < \infty$ and equality (3.1) implies that the crucial hypothesis (1.5) of the main theorem is satisfied. □

Theorem 3.2 (Local version)**.** *Let f be a C^2 $\varphi-$harmonic mapping of the unit disc U onto the $C^{2,\alpha}$ Jordan domain D having continuous extension $\tilde f$ to the boundary such that $\tilde f(\partial U) = \partial D$. If f is quasiconformal in some neighborhood of a point $z_0 \in T = \partial U$ and $(\ln\varphi)'$ is bounded in some neighborhood of $w_0 = f(z_0)$, then f has bounded partial derivatives and in particular, it is a Lipschitz mapping in a neighborhood of the point z_0.*

Proof. Let be $r > 0$ such that f is q.c. in $U_0 = D(z_0, r) \cap U$. Then $\gamma_0 = f(T \cap D(z_0, r))$ is a $C^{2,\alpha}$ Jordan arc in ∂D containing w_0. Now following the proof of Theorem 1.3, we obtain that the function f has bounded partial derivatives near the arc $\gamma = f(T \cap \bar D(z_0, r/2))$ and so in some neighborhood of the point z_0. □

Definition 3.3. A function χ which is of class C^1 and satisfies the inequality $|\bar\partial\chi| \le M|\chi|$ in a domain D is said to be approximately analytic in D with the constant M.

If a φ-metric satisfies the hypothesis $M = ||(\log\varphi)'||_\infty < \infty$ on a domain D, then it is approximately analytic. The hypothesis implies that $|\varphi'| \le M|\varphi|$ on D. Hence, since $|\varphi|_z \le |\varphi'|$ and $2\rho_z\rho = |\varphi|_z$, it follows $2\rho_z\rho = |\varphi|_z \le |\varphi'| \le M|\varphi| = M\rho^2$ on D. Thus the metric is approximately analytic in D with the constant $M/2$.

The following theorem, concerning approximately analytic metric, is a generalization of Theorem 3.1.

Theorem 3.4. *Let f be a C^2 $\rho-$harmonic mapping of the unit disc U onto the $C^{2,\alpha}$ Jordan domain D. If the metric ρ is approximately analytic in D and f is quasiconformal, then f has bounded partial derivatives; and, in particular, it is Lipschitz mapping.*

The proof of the Theorem 3.4 follows directly from Theorem 1.3(the main result), using the fact that the equation $|\bar\partial\chi| = |\partial\chi|$ holds for all real functions χ. The following theorem can be proved in the same way as Theorem 3.2.

Theorem 3.5 (Local version)**.** *Let f be a C^2 $\rho-$harmonic mapping of the unit disc U onto the $C^{2,\alpha}$ Jordan domain D having continuous extension $\tilde f$ to the boundary such that $\tilde f(\partial U) = \partial D$. If f is quasiconformal in some neighborhood of a point $z_0 \in T = \partial U$, and the metric ρ is approximately analytic in some neighborhood of $w_0 = f(z_0)$, then f has bounded partial derivatives, and in particular it is a Lipschitz mapping in a neighborhood of the point z_0.*

The harmonic and q.c. mappings between Riemann surfaces

Similarly as in the case of domains of complex plane we define quasiconformal mapping and harmonic mapping $f : R \mapsto S$ between the Riemann surfaces R and S with the metrics ϱ and ρ respectively.

If f is a harmonic mapping then

$$\varphi dz^2 = \rho \circ f \, f_z \, \overline{f}_{\bar{z}} \, dz^2 \tag{3.2}$$

is a holomorphic quadratic differential on R, and we say that φ is the *Hopf differential* of f and we write $\varphi =$Hopf(f).

Lemma 3.6. *Let (S_1, ρ_1) and (S_2, ρ_2) and (R, ρ) be three Riemann surfaces. Let g be an isometric transformation of the surface S_1 onto the surface S_2:*

$$\rho_1(\omega)|d\omega|^2 = \rho_2(w)|dw|^2, \ w = g(\omega).$$

Then $f : R \mapsto S_1$ is ρ_1- harmonic if and only if $g \circ f : R \mapsto S_2$ is ρ_2-harmonic. In particular, if g is an isometric self-mapping of S_1, then f is ρ_1- harmonic if and only if $g \circ f$ is ρ_1- harmonic.

Proof. If f is a harmonic map then $\varphi dz^2 = \rho \circ f \, p \, \overline{q} \, dz^2$ is a holomorphic quadratic differential in R, i.e., the mapping $\rho \circ f \, p \, \overline{q}$ is analytic near to the parameter $z = z(\zeta)$, $\zeta \in R$. Let $\omega = f(z)$, $F = g \circ f$, $P = (g \circ f)_z$ and $Q = (g \circ f)_{\bar{z}}$. Then $P = g'(\omega) \cdot p$ and $Q = g'(\omega) \cdot q$. Since $\rho_1(\omega) = \rho_2(w)|g'(\omega)|^2$, we obtain

$$\rho_2 \circ F \, P\bar{Q} = \rho_2 \circ g \circ f \cdot |g'(\omega)|^2 p\bar{q} = \rho_1 \circ f p\bar{q}.$$

Hence $\varphi_1 =$Hopf$(g \circ f)$ is a holomorphic differential, i.e., $g \circ f$ is harmonic w.r. to the metric ρ_2. $\square$

Instead of an arbitrary Riemann surface we consider here only the Riemann sphere. Note that most of the arguments work for an arbitrary compact Riemann surface.

The metric ρ defined on $S^2 = \overline{\mathbb{C}}$ by

$$\rho|dz|^2 = \frac{4|dz|^2}{(1 + |z|^2)^2}$$

we call *the spherical metric.* The corresponding distance function is

$$d_S(z, w) = \frac{2|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}}, \ d_S(z, \infty) = \frac{2}{\sqrt{(1 + |z|^2)}}. \tag{3.3}$$

We can verify that the orientation preserving isometries of Riemann sphere S^2 w.r. to the spherical metric are described by Möbius transformations of the form

$$g(z) = \frac{az + b}{\bar{a} - \bar{b}z}, \ a, b \in \mathbb{C}, \ |a|^2 + |b|^2 \neq 0. \tag{3.4}$$

The Euler-Lagrange equation for spherical harmonic mappings is

$$f_{z\bar{z}} - \frac{2\bar{f}}{1 + |f|^2} \, f_z \cdot f_{\bar{z}} = 0. \tag{3.5}$$

It is easy to verify that the spherical density is approximately analytic in $\mathbb{C}$ with the constant 1; more precisely one can verify

$$\frac{\rho_{\overline{z}}}{\rho} = \frac{2z}{1+|z|^2}.$$

If f is a diffeomorphism of the Riemann sphere (or of a compact Riemann surface M) onto itself, then f is quasi-isometry w.r. to the corresponding metric and consequently, it is quasiconformal.

A natural question is what we can say for harmonic q.c. diffeomorphisms defined in some sub-domain of the Riemann sphere.

Using Theorem 3.4, Lemma 3.6 and the isometries defined by (3.4) we can prove:

Proposition 3.7. *Let the domains $\Omega, D \subset \overline{\mathbb{C}}$ have $C^{1,\alpha}$ and $C^{2,\alpha}$ Jordan boundary on $S^2 = \overline{\mathbb{C}}$, respectively. Then any q.c. spherical harmonic diffeomorphism of Ω onto D is Lipschitz w.r. to the spherical metric.*

4. Representation of $\varphi-$harmonic mappings

If f is $\varphi-$harmonic, and ϕ is so called natural parameter defined by φ then the mapping $F = \phi \circ f$ is an euclidean harmonic. Application of Theorem K (see the introduction) to $F = \phi \circ f$ leads to Theorem 4.4(the main result of this section), which shows that Theorem 3.1 holds for more general domains.

Recall that if f is $\varphi-$*harmonic*, then it satisfies the equation (3.1). If $\varphi(w_0) \neq 0$, then there is a neighborhood V of w_0 such that there is a branch $\sqrt{\varphi}$ in V such that

$$\phi = \int \sqrt{\varphi(z)}\, dz \tag{4.1}$$

is conformal on V.

In this setting, we refer to $\phi = \int \sqrt{\varphi(z)}\, dz$ as a natural parameter on V.

Theorem 4.1. *If f is $\varphi-$harmonic and ψ is conformal on the co-domain of f, then the mapping $F = \psi \circ f$ satisfies the following equation:*

$$F_{z\bar{z}} = \left[\frac{\psi''(w)}{\psi'(w)^2} - \frac{\varphi'(w)}{2\psi'(w)\cdot\varphi(w)}\right]\cdot F_z \cdot F_{\bar{z}}, \tag{4.2}$$

where $w = f(z)$.

Proof. Since ϕ is analytic we have that $F_z = \psi'(w)\cdot f_z$ and that $F_{\bar{z}} = \psi'(w)\cdot f_{\bar{z}}$. Hence $F_{z\bar{z}} = \psi''(w) f_z f_{\bar{z}} + \psi'(w) f_{z\bar{z}}$. On the other hand, f is $\varphi-$harmonic and therefore:

$$f_{z\overline{z}} = -\frac{1}{2}\frac{\varphi'}{\varphi}\circ f \cdot f_z\, f_{\bar{z}}.$$

Combining those facts, we obtain (4.2).

□

Notice that if $\varphi = 1$, then φ-metric is reduced to Euclidean metric; so if f is an Euclidean harmonic mapping, then

$$F_{z\bar{z}} = \frac{\psi''}{\psi'^2} F_z \cdot F_{\bar{z}} \,. \tag{4.3}$$

Corollary 4.2. *Let φ be an analytic function such that there exists a branch of $\int \sqrt{\varphi(z)}\,\mathrm{dz}$ in some domain D. If $f : \Omega \mapsto D$ is $\varphi-$harmonic and*

$$\phi = \int \sqrt{\varphi(z)}\, dz,$$

then the mapping $F = \phi \circ f$ is harmonic with respect to the Euclid metric.

Proof. We easily obtain

$$\frac{\phi''(w)}{\phi'(w)^2} - \frac{\varphi'(w)}{2\phi'(w)\cdot\varphi(w)} = 0.$$

It follows from (4.2) that $F_{z\bar{z}} \equiv 0$. Hence F is harmonic. □

Using (4.3) we obtain:

Corollary 4.3. *Let h be an Euclidean harmonic mapping, let ψ be conformal on the co-domain of h; and let $\varphi = ((\psi^{-1})')^2$. Then the mapping $\hat{h} = \psi \circ h$ is φ- harmonic.*

Recall, if f is $\varphi-$harmonic, and ϕ is natural parameter defined by φ then the mapping $F = \phi \circ f$ is Euclidean harmonic. Applying Theorem K (see the introduction and also [14, Theorem 3.1]) to $C^{1,\alpha}$ co-domain $D' = \phi(D)$ and Euclidean harmonic mapping $F = \phi \circ f$ (note that ϕ is not 1-1 in general), we can prove that Theorem 3.1 holds for more general domains.

Theorem 4.4. *Let f be a $\varphi-$harmonic mapping of the $C^{1,\alpha}$ domain Ω onto the $C^{1,\alpha}$ Jordan domain D. If $M = ||(\log\varphi)'||_\infty < \infty$ and f is quasi-conformal, then f has bounded partial derivatives and in particular, it is a Lipschitz mapping.*

Assume that $\varphi(z) \neq 0$ and that the natural parameter

$$\phi(z) = \int \sqrt{\varphi(z)}\, dz$$

is well defined on a domain D; and let ϕ maps D onto the convex domain $D' = \phi(D)$.

By the definition of φ- metric, we have that:

$$d(z, w) = \inf_{z,w\in\gamma\subset D} \int_\gamma \sqrt{|\varphi(\zeta)|}|d\zeta|.$$

Since

$$\sqrt{|\varphi(\zeta)|}|d\zeta| = |d(\phi(\zeta))|,$$

setting $A = \phi(z)$, $B = \phi(w)$ and $\xi = \phi(\zeta)$, by the chain rule we obtain that

$$d(z,w) = \inf_{A,B\in\gamma'\subset D'} \int_{\gamma'} |d\xi|,$$

where $D' = \phi(D)$.

Now it is clear that the segment $[A,B]$ that belongs to D' (because D' is convex), is the curve that minimizes the previous functional. Hence $d(z,w) = |A-B| = |\phi(z)-\phi(w)|$. Thus we have proved the following proposition:

Proposition 4.5. *If $D' = \phi(D)$ is convex, then ϕ transforms the φ-metric to Euclidean metric; i.e. the distance function defined by φ- metric is given by the formula*

$$d(z,w) = |\phi(z)-\phi(w)|\,.$$

Example 4.6. Let $\varphi_0(w) = \frac{1}{(w-c_0)^2}$ and let us consider the harmonic maps between two domains Ω and D with respect to the following metric density on D:

$$\rho_0(w) = |\varphi_0(w)| = \frac{1}{|w-c_0|^2},\ w\in D, \tag{4.4}$$

where $c_0 \notin \overline{D}$ is a given point. If $D' = \log(D-\{c_0\})$ is a convex domain, then the metric defined by the metric density (4.4) is

$$d_0(z,w) = \left|\log\frac{z-c_0}{w-c_0}\right|.$$

It is easy to verify that the conformal mappings A:

$$A(z) = c_0 + re^{i\alpha(\varepsilon-1)/2}(z-c_0)^{\varepsilon},\ r\in\mathbb{R},\ \varepsilon=\pm1, \tag{4.5}$$

describe the orientation preserving isometries of the domain $D_\alpha = \mathbb{C}\setminus\{c_0 + te^{i\alpha}, t\in\mathbb{R}^+\}$, with respect to the metric d_0 given by (4.4).

Let f be φ_0-harmonic between Ω and D, where $D\subset D_\alpha$ for some α. The natural parameter is $\phi_0(w) = \pm\log(w-c_0)$. Hence, as an application of the Corollary 4.2, we obtain that $F(z) = \log(f(z)-c_0)$ is a harmonic function defined on the simply connected domain Ω. Hence we have

$$f(z)-c_0 = e^{g_0(z)+\overline{h_0(z)}} = g_1(z)\cdot\overline{h_1(z)} = \big(\sqrt{c_0}-\frac{1}{\sqrt{c_0}}g(z)\big)\cdot\big(-\overline{\sqrt{c_0}}+\frac{1}{\overline{\sqrt{c_0}}}\overline{h(z)}\big),$$

which yields the representation:

$$f(z) = g(z) + \overline{h(z)} - {c_0}^{-1}g(z)\cdot\overline{h(z)}, \tag{4.6}$$

where g and h are analytic mappings, which map Ω into $\mathbb{C}\setminus\{c_0\}$.

It is easy to see that the family of mappings defined by (4.6) is closed under transformations given by (4.5) (see Lemma 3.6).

The above example provides the motivation for the following result.

Theorem 4.7. *Let g and h be analytic functions and let $f = g+\overline{h}-{c_0}^{-1}g\overline{h}$, $c_0 \neq 0$, be a diffeomorphism of the $C^{1,\beta}$ domain Ω onto the $C^{1,\alpha}$ Jordan domain D such that $c_0 \in \overline{\mathbb{C}} \setminus \overline{D}$. If f is q.c. mapping, then it has bounded partial derivatives and the analytic functions g' and h' are bounded.*

Proof. The case $c_0 = \infty$ is proved by Theorem 4.4 and therefore we can assume that $c_0 \neq \infty$. Put

$$g_1 = \sqrt{c_0} - \frac{1}{\sqrt{c_0}}g \quad \text{and} \quad h_1 = -\overline{\sqrt{c_0}} + \frac{1}{\overline{\sqrt{c_0}}}h.$$

Then $f-c_0 = g_1 \cdot \overline{h}_1$. Since $f(z) \neq c_0$ it follows that $h_1(z) \neq 0$ and $g_1(z) \neq 0$. Therefore we can take the mapping $F = \log(f - c_0)$ which can be written as $F = \log g_1 + \overline{\log h_1}$ on Ω. Hence F is a harmonic mapping of Ω onto $C^{1,\alpha}$ domain $D' = \log(D - c_0)$. We obtain from Theorem 4.4 that there exists a constant M such that

$$\left|\frac{h_1'}{h_1}\right|^2 + \left|\frac{g_1'}{g_1}\right|^2 < M. \tag{4.7}$$

Thus $(\log h_1)'$ is bounded on Ω and consequently $\log h_1$ has a continuous extension to the boundary of Ω. Thus h_1 has a continuous and non-vanishing extension to $\overline{\Omega}$. The same holds for G.

Now, by 4.7, we obtain that h_1' and g_1' are bounded mappings. Thus h' and g' are bounded. □

Example 4.8. A harmonic mapping u with respect to the hyperbolic metric on the unit disk satisfies the following equation

$$u_{z\bar{z}} + \frac{2\bar{u}}{1-|u|^2}\, u_z \cdot u_{\bar{z}} = 0.$$

As far as we know this equation cannot be solved using the known methods of PDE; however, we can produce some examples; more precisely, we characterize real hyperbolic harmonic mappings.

Let

$$\varphi_1(w) = \frac{4}{(1-w^2)^2}.$$

Using a natural parameter, i.e. a branch of $\phi_1(z) = \log\frac{z+1}{z-1} = 2\,arc\tanh z$, one can verify that f is φ_1-harmonic if and only if $f = \tanh g$, where g is Euclidean harmonic. Since the metric $\rho = |\varphi(w)|$ coincides with the Poincaré metric

$$\lambda = \frac{4}{(1-|w|^2)^2}$$

for real w we obtain that f *is real λ- harmonic (hyperbolic harmonic) if and only if $f = 2\tanh g$, where g is real Euclidean harmonic.* Since the mappings

$$w = e^{i\varphi}\frac{z-a}{1-\bar{a}z}, \quad a \in \mathbb{U},$$

are the isometries of Poincaré disc, because of Lemma 3.6, we obtain the following claim: If h is real harmonic defined on some domain Ω, then the function

$$w = e^{i\varphi}\frac{\tanh(h(z)) - a}{1 - \bar{a}\tanh(h(z))} \quad (|a| < 1) \tag{4.8}$$

is harmonic w.r. to hyperbolic metric . Note that the mappings given by (4.8) have the rank 1 and they map Ω into circular arcs orthogonal on the unit circle $\mathbb{T}$.

Moreover, if a circle S orthogonal on the unit circle is given and $\Lambda = S \cap \mathbb{T}$, we can use (4.8) to describe all λ- harmonic mappings between Ω and Λ.

Acknowledgment. We would like to express our gratitude to the referee for useful comments and suggestions related to this paper.

Some part of this manuscript was communicated by the second author at The X-th Romanian-Finnish Seminar, August 14-19, 2005, Cluj-Napoca, Romania and at Helsinki-Turku Seminar (during the visit of the second author to the Universities of Helsinki and Turku in October 2005). Participants of those seminars gave useful comments; in particular, it was suggested by Olli Martio that Theorem MK1-2 may have several dimension analogy (more details about it will appear in [21]) .

References

[1] L. Ahlfors, *Lectures on quasiconformal mappings*, Van Nostrand Princeton, N.J., 1966.
[2] I. Anić, V. Marković and M. Mateljević, *Uniformly bounded maximal φ-discs and Bers space and Harmonic maps*, Proc. Amer. Math. Soc. **128** (2000) 2947-2956
[3] A. Beurling and L. Ahlfors: *The boundary correspondence under quasiconformal mappings*, Acta Math. 96 (1956), 125-142.
[4] J. Clunie and T. Sheil-Small, *Harmonic univalent function*, Ann. Acad. Sci. Fenn. A I Math. **9** 1984, 3-25.
[5] P. Duren, *Harmonic mappings in the plane*, Cambridge Univ. Press, 2004.
[6] Z-C. Han, *Remarks on the geometric behavior of harmonic maps between surfaces*, Elliptic and parabolic methods in geometry. Proceedings of a workshop, Minneapolis, May 23-27, 1994., Wellesle.
[7] P. Li, L. Tam and J. Wang, *Harmonic diffeomorphisms between hyperbolic Hadamard manifolds*, to appear in Jour. Geom. Anal.
[8] G. M. Goluzin, *Geometric function theory* , Nauka Moskva 1966 (Russian).
[9] E. Heinz, *On certain nonlinear elliptic differential equations and univalent mappings,* J. d' Anal. 5, 1956/57, 197-272.
[10] D. Kalaj, *Harmonic functions and harmonic quasiconformal mappings between convex domains,* Thesis, Beograd 2002.
[11] D. Kalaj, *On harmonic diffeomorphisms of the unit disc onto a convex domain.* Complex Variables, Theory Appl. 48, No.2, (2003), 175-187.
[12] D. Kalaj, *Quasiconformal harmonic functions between convex domains*, Publ. Inst. Math., Nouv. Ser. 76(90), 3-20 (2004).
[13] D. Kalaj and M. Pavlović, *Boundary correspondence under harmonic quasiconformal homeomorfisms of a half-plane,* Ann. Acad. Sci. Fenn., Math. 30, No.1, (2005) 159-165.
[14] D. Kalaj, *Quasiconformal harmonic functions between Jordan domains*, preprint.

[15] O. Kellogg, *On the derivatives of harmonic functions on the boundary,* Trans. Amer. Math. Soc. 33 (1931), 689-692.

[16] M. Knežević and M. Mateljević: *On q.c. and harmonic mapping and the quasi-isometry*, preprint.

[17] O. Martio, *On harmonic quasiconformal mappings*, Ann. Acad. Sci. Fenn., Ser. A I 425 (1968), 3-10.

[18] M. Mateljević, *Note on Schwarz lemma, curvature and distance,* Zbornik radova PMF **13** (1992) 25-29

[19] M. Mateljević, *Ahlfors-Schwarz lemma and curvature,* Kragujevac Journal of Mathematics (Zbornik radova PMF),Vol **25,** (2003) 155-164.

[20] M. Mateljević, *Dirichlet's principle, distortion and related problems for harmonic mappings*, Publication l'Inst Math -Belgrade, nouvelle serie 75(89), (2004) 147-171 (special number Quasiconformal and Harmonic mappings, special guest editor M. Mateljević).

[21] M. Mateljević, Distortion of harmonic functions and harmonic quasiconformal quasi-isometry, submited to Proceedings of The X-th Romanian-Finnish Seminar, August 14-19, 2005, Cluj-Napoca.

[22] M. Pavlović, *Boundary correspondence under harmonic quasiconformal homeomorfisms of the unit disc*, Ann. Acad. Sci. Fenn., Vol 27, (2002) 365-372.

[23] C. Pommerenke, *Boundary behavour of conformal maps,* Springer-Verlag, New York, 1991

[24] L. Tam and T. Wan, *Quasiconformal harmonic diffeomorphism and universal Teichmüler space*, J. Diff. Geom. **42** (1995) 368-410.

[25] L. Tam and T. Wan, *On quasiconformal harmonic maps*, Pacific J Math, V 182 (1998), 359-383.

[26] L. Tam and T. Wan, *Harmonic diffeomorphisms into Cartan-Hadamard surfaces with prescribed Hopf differentials*, Comm. Anal. Geom. **4** (1994) 593-625.

[27] A. Zygmund, *Trigonometric Series I.* Cambrige University Press, 1958.

[28] T. Wan, Constant mean curvature surface, harmonic maps, and universal Teichmüller space, J. Diff. Geom. 35 (1992) 643-657.

[29] Y. Shi, L. Tam and T. Wan *Harmonic maps on hyperbolic spaces with singular boundary value* March, 1998.

University of Montenegro, Faculty of Natural Sciences and Mathematics, Cetinjski put b.b. 8100 Podgorica, Montenegro
E-mail address: davidk@cg.yu

University of Belgrade, Faculty of Mathematics, Studentski trg 16, 11000 Belgrade, Serbia
E-mail address: miodrag@matf.bg.ac.yu