HARMONIC MAPS BETWEEN ANNULI ON RIEMANN SURFACES

DAVID KALAJ

Abstract. Let $\rho_\Sigma = h(|z|^2)$ be a metric in a Riemann surface Σ, where h is a positive real function. Let $\mathcal{H}_{r_1} = \{w = f(z)\}$ be the family of univalent ρ_Σ harmonic mapping of the Euclidean annulus $A(r_1, 1) := \{z : r_1 < |z| < 1\}$ onto a proper annulus A_Σ of the Riemann surface Σ, which is subject of some geometric restrictions. It is shown that if A_Σ is fixed, then $\sup\{r_1 : \mathcal{H}_{r_1} \neq \emptyset\} < 1$. This generalizes the similar results from Euclidean case. The cases of Riemann and of hyperbolic harmonic mappings are treated in detail. Using the fact that the Gauss map of a surface with constant mean curvature (CMC) is a Riemann harmonic mapping, an application to the CMC surfaces is given (see Corollary 3.2). In addition some new examples of hyperbolic and Riemann radial harmonic diffeomorphisms are given, which have inspired some new J. C. C. Nitsche type conjectures for the class of these mappings.

Contents

1. Introduction and preliminaries

It is well known that an annulus $A(r_1, 1) := \{z : r_1 < |z| < 1\}$ can be mapped conformally onto an annulus $A(\varrho, 1) = \{w : \varrho < |w| < 1\}$ if and only if $r_1 = \varrho$, i.e. if they have the same modulus. If f is K-quasiconformal, then $r_1^K \leq \varrho \leq r_1^{1/K}$ [13, p. 38]. However, if f is neither conformal nor quasiconformal, then ϱ is possibly zero, as with the harmonic mapping

$$f(z) = \frac{z - r_1^2/\bar{z}}{1 - r_1^2};$$

which can easily be shown to map $A(r_1, 1)$ univalently onto the punctured unit disc $A(0, 1)$.

1991 *Mathematics Subject Classification.* 58E20,30F45.
Key words and phrases. Harmonic maps, Riemann surfaces, Modulus of annuli, Gauss map.

J. C. C. Nitsche [18] by considering the complex-valued univalent harmonic functions

$$f(z) = \frac{r_1\varrho - 1}{r_1^2 - 1} z + \frac{r_1^2 - r_1\varrho}{r_1^2 - 1}\frac{1}{\overline{z}},$$

showed that an annulus $r_1 < |z| < 1$ can then be mapped onto any annulus $\varrho < |w| < 1$ with

$$\varrho \leq \frac{2r_1}{1+r_1^2}. \tag{1.1}$$

J. C. C. Nitsche conjectured that, condition (1.1) is necessary as well.

He also showed that $\varrho \leq \varrho_0$ for some constant $\varrho_0 = \varrho_0(r_1) < 1$. Thus although the annulus $r_1 < |z| < 1$ can be mapped harmonically onto a punctured disk, it cannot be mapped onto any annulus that is "too thin". A. Lyzzaik [15] recently gave a quantitative bound for ϱ_0, showing that $\varrho_0 \leq s$ if the annulus $r_1 < |z| < 1$ is conformally equivalent to the Grötzsch domain consisting of the unit disk minus the radial slit $0 \leq x \leq s$. Weitsman in [26] showed that

$$\varrho \leq \frac{1}{1 + \frac{1}{2}(r_1 \log r_1)^2},$$

an improvement on Lyzzaik's result when ϱ is near 1. The author in [10] improved Weitsman's bound for all ϱ showing that

$$\varrho \leq \frac{1}{1 + \frac{1}{2}\log^2 r_1}.$$

Very recently, in [8] is proved the Nitsche conjecture when the domain annulus is not too wide; explicitly, when $\log\frac{1}{r_1} \leq 3/2$. For general $A(r_1, 1)$ the conjecture is proved under the additional assumption that either h or its normal derivative have vanishing average on the inner boundary circle.

In general, however, this remains an attractive unsettled problem. See also [2], [3], [21], [11] and [9] for a related topic.

Zheng-Chao Han in [24] obtained a similar result for hyperbolic harmonic mappings from the unit disk onto the half-plane.

In this paper we consider the situation when the metric is negatively curved or is positively curved uniformly. Notice that the Euclidean metric has a vanishing Gauss curvature.

By $\mathbb{U}$ we denote the unit disk $\{z : |z| < 1\}$, by $\overline{\mathbb{C}}$ is denoted the extended complex plane. Let Σ be a Riemannian surface over a domain C of the complex plane or over $\overline{\mathbb{C}}$ and let $p : C \mapsto \Sigma$ be a universal covering. Let ρ_Σ be a conformal metric defined in the universal covering domain C or in some chart D of Σ. It is well-known that C can be one of three sets: $\mathbb{U}$, $\mathbb{C}$ and $\overline{\mathbb{C}}$. Then the distance function is defined by

$$d(a, b) = \inf_{a,b\in\gamma} \int_0^1 \rho_\Sigma(\tilde{\gamma}(t))|\tilde{\gamma}'(t)|dt,$$

where $\tilde{\gamma}$, $\tilde{\gamma}(0) = 0$, is a lift of γ, i.e. $p(\tilde{\gamma}(t)) = \gamma(t)$, $\gamma(0) = a$, $\gamma(1) = b$.

The Gauss curvature of the surface (and of the metric ρ_Σ) is given by

$$K = -\frac{\Delta \log \rho_\Sigma}{\rho_\Sigma^2}.$$

In this paper we will consider those surfaces Σ, whose metric have the form

$$\rho_\Sigma(z) = h(|z|^2),$$

defined in some chart D of Σ (not necessarily in the whole universal covering surface). Here h is an positive twice differentiable function. We call these metrics radial symmetric.

1.1. **Riemann surfaces with radially symmetric metrics.** A Riemann surface does not come equipped with any particular Riemannian metric. However, the complex structure of the Riemann surface does uniquely determine a metric up to conformal equivalence. (Two metrics are said to be conformally equivalent if they differ by multiplication by a positive smooth function.) Conversely, any metric on an oriented surface uniquely determines a complex structure, which depends on the metric only up to conformal equivalence. Complex structures on an oriented surface are therefore in one-to-one correspondence with conformal classes of metrics on that surface. Within a given conformal class, one can use conformal symmetry to find a representative metric with convenient properties. In particular, there is always a complete metric with constant curvature in any given conformal class. We begin by the case of metrics with negative curvature.

1.1.1. *Hyperbolic metrics.* For every hyperbolic Riemann surface, the fundamental group is isomorphic to a Fuchsian group, and thus the surface can be modeled by a Fuchsian model $\mathbb{U}/\Gamma$, where $\mathbb{U}$ is the unit disk and Γ is the Fuchsian group ([1]). If Ω is a hyperbolic region in the Riemann sphere $\overline{\mathbb{C}}$; i.e., Ω is open and connected with its complement $\Omega^c := \overline{\mathbb{C}} \setminus \Omega$ possessing at least three points. Each such Ω carries a unique maximal constant curvature -1 conformal metric $\lambda|dz| = \lambda_\Omega(z)|dz|$ referred to as the Poincaré hyperbolic metric in Ω. The domain monotonicity property, that larger regions have smaller metrics, is a direct consequence of Schwarz's Lemma. Except for a short list of special cases, the actual calculation of any given hyperbolic metric is notoriously difficult.

By the formula

$$\rho_\Sigma(z) = h(|z|^2),$$

we obtain that the Gauss curvature is given by

$$K = \frac{4(|z|^2 h'^2 - |z|^2 h h'' - h h')}{h^4}.$$

Setting $t = |z|^2$, we obtain that

$$K = -\frac{1}{h^2}\left(\frac{4th'(t)}{h}\right)'. \tag{1.2}$$

As $K \le 0$ it follows that

$$\left(\frac{4th'(t)}{h}\right)' \ge 0.$$

Therefore the function

$$\frac{4th'(t)}{h}$$

is increasing, i.e.

$$t \ge s \Rightarrow \frac{4th'(t)}{h(t)} \ge \frac{4sh'(s)}{h(s)}, \tag{1.3}$$

and in particular

$$t \geq 0 \Rightarrow \frac{4th'(t)}{h(t)} \geq 0. \tag{1.4}$$

In this case we obtain that h is an increasing function.
The examples of hyperbolic surfaces are:

(1) The Poincaré disk $\mathbb{U}$ with the hyperbolic metric
$$\lambda = \frac{2}{1-|z|^2}.$$
(2) The punctured hyperbolic unit disk $\Delta = \mathbb{U} \setminus \{0\}$. The linear density of the hyperbolic metric on Δ is
$$\lambda_\Delta = \frac{1}{|z| \log \frac{1}{|z|}}.$$
(3) The hyperbolic annulus $A(1/R, R)$, $R > 1$. The hyperbolic metric is given by
$$h_R(|z|^2) = \lambda_R(z) = \frac{\pi/2}{|z| \log R} \sec(\frac{\pi}{2} \frac{\log |z|}{\log R}).$$
In all these cases the Gauss curvature is $K = -1$.

1.1.2. *Riemann metrics.* In the case of the Riemann sphere, the Gauss-Bonnet theorem implies that a constant-curvature metric must have positive curvature K. It follows that the metric must be isometric to the sphere of radius $1/\sqrt{K}$ in $\mathbb{R}^3$ via stereographic projection. In the z-chart on the Riemann sphere, the metric with $K = 1$ is given by

$$ds^2 = h_R^2(|z|^2)|dz|^2 = \frac{4|dz|^2}{(1+|z|^2)^2}.$$

Another important case is Hamilton cigar soliton or in physics is known as *Wittens's black hole.* It is a Kähler metric defined on $\mathbb{C}$.

$$ds^2 = h^2(|z|^2)|dz|^2 = \frac{|dz|^2}{1+|z|^2}.$$

The Gauss curvature is given by

$$K = \frac{2}{1+|z|^2}.$$

In both these cases $K > 0$. This means that

$$\left(\frac{4th'(t)}{h}\right)' \leq 0.$$

Therefore the function

$$\frac{4th'(t)}{h}$$

is decreasing i.e.

$$t \geq s \Rightarrow \frac{4th'(t)}{h(t)} \leq \frac{4sh'(s)}{h(s)}. \tag{1.5}$$

In this case we obtain that h is a decreasing function.

1.1.3. *The definition of harmonic mappings.* Let (M,σ) and (N,ρ) be Riemann surfaces with metrics σ and ρ, respectively. If $f:(M,\sigma)\to(N,\rho)$ is a C^2, then f is said to be harmonic with respect to ρ if

$$f_{z\bar z}+(\log\rho^2)_w\circ f f_z\, f_{\bar z}=0, \tag{1.6}$$

where z and w are the local parameters on M and N respectively.

Also f satisfies (1.6) if and only if its Hopf differential

$$\Psi=\rho^2\circ f f_z\overline{f_{\bar z}} \tag{1.7}$$

is a holomorphic quadratic differential on M.

For $g:M\mapsto N$ the energy integral is defined by

$$E[g,\rho]=\int_M \rho^2\circ f(|\partial g|^2+|\bar\partial g|^2)dV_\sigma. \tag{1.8}$$

where ∂g, and $\bar\partial g$ are the partial derivatives taken with respect to the metrics ϱ and σ, and dV_σ is the volume element on (M,σ). Assume that energy integral of f is bounded. Then f is harmonic if and only if f is a critical point of the corresponding functional where the homotopy class of f is the range of this functional. For this definition and the basic properties of harmonic map see [22].

Using the fact that the function defined in (1.7) is holomorphic, the following well known lemma can be proved (see e.g. [12]).

Lemma 1.1. *Let (S_1,ρ_1) and (S_2,ρ_2) and (R,ρ) be three Riemann surfaces. Let g be an isometric transformation of the surface S_1 onto the surface S_2:*

$$\rho_1(\omega)|d\omega|^2=\rho_2(w)|dw|^2,\ w=g(\omega).$$

Then $f:R\mapsto S_1$ is ρ_1- harmonic if and only if $g\circ f:R\mapsto S_2$ is ρ_2- harmonic. In particular, if g is an isometric self-mapping of S_1, then f is ρ_1- harmonic if and only if $g\circ f$ is ρ_1- harmonic.

The rest of this paper is organized as follows. In Section 2, using Lemma 2.1, which deals with local character of geodesic lines, we show that an annulus $A(r_1,1)$ of the Euclidean plane can be mapped by means of harmonic mappings to a fixed annulus of a Riemann surface only if r_1 is not so close to 1, (Theorem 2.3). The annulus of the Riemann surface has a special character, see the formulation of Theorem 2.3, but this can be apply to all annuli that are isometric to this annulus (see Lemma 1.1). These isometries are well-known in the case of Riemann sphere and of Hyperbolic plane. In Section 3 this problem is treated for Riemann harmonic mappings in details. Section 4 deals with the hyperbolic harmonic mappings. Moreover some examples of radial harmonic mappings are given and these mappings have inspired some new J. C. C. Nitsche type conjectures.

2. The main results

In this section we will consider an annulus of a Riemann surface Σ whose boundary components are homotopic to a point $0\in\Sigma$. Similarly, the case of annuli generated by an oriented Jordan curve γ that is not homotopic to a point can be considered. For the definition of ring domains of this type see [23, p. 9–11]. In Theorem 4.1 a special case of a ring domain associated with Jordan curve that is not homotopic to a point of Σ is considered.

We begin by this useful lemma.

Lemma 2.1. *If the metric ρ_Σ in a chart D of a Riemann surface Σ is given by $\rho_\Sigma(z) = h(|z|^2)$, then the intrinsic distance of $lz, z \in D$, $l < 1$, with $[lz, z] \subset D$, is given by*

$$d_\Sigma(lz, z) = \int_{l|z|}^{|z|} h(t^2)dt. \tag{2.1}$$

In particular, if $z \in D$ and if $[0, z] \in D$ then $[0, z]$ is a geodesic in D with respect to the metric ρ_Σ. A similar formula holds for $l > 1$

Proof. To prove this we do as follows.

Since $g_{11} = g_{22} = h^2(|z|^2)$, and $g_{12} = g_{21} = 0$, using the formula

$$\Gamma^i{}_{k\ell} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial x^\ell} + \frac{\partial g_{m\ell}}{\partial x^k} - \frac{\partial g_{k\ell}}{\partial x^m}\right) = \frac{1}{2}g^{im}(g_{mk,\ell} + g_{m\ell,k} - g_{k\ell,m}),$$

where the matrix (g^{jk}) is an inverse of the matrix (g_{jk}), we obtain that the Christoffel symbols of our metric are given by:

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = \frac{h_x}{h}, \tag{2.2}$$

$$\Gamma^2_{22} = \Gamma^1_{12} = \Gamma^1_{21} = \frac{h_y}{h}, \tag{2.3}$$

$$\Gamma^2_{11} = -\frac{h_x}{h}, \quad \Gamma^1_{22} = -\frac{h_y}{h}. \tag{2.4}$$

The geodesic equations are given by:

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \ , \lambda = 1, 2.$$

In view of (2.2), (2.3) and (2.4) we obtain the system:

$$\ddot{x} + 2\frac{xh'}{h}\dot{x}^2 + 4\frac{yh'}{h}\dot{x}\dot{y} - 2\frac{xh'}{h}\dot{y}^2 = 0, \tag{2.5}$$

$$\ddot{y} - 2\frac{yh'}{h}\dot{x}^2 + 4\frac{xh'}{h}\dot{x}\dot{y} + 2\frac{yh'}{h}\dot{y}^2 = 0. \tag{2.6}$$

Assume, first that $[l|z|, |z|] \subset D$. Denote the geodesic curve joining the points $|z|$ and $|z|l$ by $c(s) := (x(s), y(s))$.

Putting $y = 0$ in (2.5) and (2.6) we obtain that x is a solution of the differential equality

$$\ddot{x} + 2\frac{xh'}{h}\dot{x}^2 = 0$$

and consequently

$$\dot{x} = \frac{C_1}{h(x^2)},$$

i.e.

$$s = C_1\int_{x_0}^{x} h(t^2)dt. \tag{2.7}$$

To determine C_1 and x_0, we use the conditions $x(0) = l|z|$, and $x(s_0) = |z|$. Inserting these conditions to (2.7) we obtain

$$s = \int_{l|z|}^{x} h(t^2)dt, \tag{2.8}$$

where

$$s_0 = \int_{l|z|}^{|z|} h(t^2)dt.$$

As the metric $h(|z|^2)|dz|$ is a rotation invariant, according to (2.8) it follows that

$$d_\Sigma(lz, z) = \inf_{lz, z\in\gamma} \int_\gamma \rho_\Sigma(z)|dz| = \int_{l|z|}^{|z|} h(r^2)dr.$$

□

Let (ρ, Θ) be geodesic polar coordinates about the point 0 of a chart D of the Riemann surface Σ with the metric $\rho_\Sigma(z) = h(|z|^2)$. Let g be the inverse of the function $s \mapsto d_\Sigma(s, 0)$. Then we have

$$\rho = \int_0^{g(\rho)} h(t^2)dt.$$

Thus

$$1 = g'(\rho) \cdot h(g^2(\rho)), \tag{2.9}$$

and

$$\frac{h'}{h} = -\frac{g''}{2gg'^2}. \tag{2.10}$$

Therefore the metric of the surface can be expressed as

$$ds^2 = d\rho^2 + h(g^2(\rho))g^2(\rho)d\Theta^2.$$

If w, is a twice differentiable, then

$$w(z) = g(\rho(z))e^{i\Theta}.$$

Now we have

$$w_x = (g'\rho_x + ig\Theta_x)e^{i\Theta},$$

$$w_y = (g'\rho_y + ig\Theta_y)e^{i\Theta},$$

and thus

$$w_{xx} = (g''\rho_x^2 + g'\rho_{xx} + 2ig'\rho_x\Theta_x + ig\Theta_{xx} - g\Theta_x^2)e^{i\Theta}, \tag{2.11}$$

$$w_{yy} = (g''\rho_y^2 + g'\rho_{yy} + 2ig'\rho_y\Theta_y + ig\Theta_{yy} - g\Theta_y^2)e^{i\Theta}, \tag{2.12}$$

and

$$w_z w_{\bar z} = \frac{1}{4}\left(w_x^2 + w_y^2\right). \tag{2.13}$$

Assume now that w is harmonic. By applying (2.11), (2.12), (2.13) and (1.6) it follows that

$$\begin{aligned}(g''|\nabla\rho|^2 &+ g'\Delta\rho + 2ig'\langle\nabla\rho, \nabla\Theta\rangle + ig\Delta\Theta - g|\nabla\Theta|^2)e^{i\Theta} \\ &+ 2\frac{h'(g(\rho)^2)g(\rho)e^{-i\Theta}}{h(g(\rho)^2)}\left(g'^2|\nabla\rho|^2 + 2ig'\langle\nabla\rho, \nabla\Theta\rangle - g^2|\nabla\Theta|^2\right)e^{2i\Theta} = 0.\end{aligned}$$

Thus

$$(g''|\nabla\rho|^2 + g'\Delta\rho + 2ig'\langle\nabla\rho,\nabla\Theta\rangle + ig\Delta\Theta - g|\nabla\Theta|^2) \\ + 2\frac{h'(g(\rho)^2)g(\rho)}{h(g(\rho)^2)}\left(g'^2|\nabla\rho|^2 + 2ig'\langle\nabla\rho,\nabla\Theta\rangle - g^2|\nabla\Theta|^2)\right) = 0.$$

Therefore

$$2g'\langle\nabla\rho,\nabla\Theta\rangle + g\Delta\Theta + 4\frac{h'(g(\rho)^2)g(\rho)g'(\rho)}{h(g(\rho)^2)}\langle\nabla\rho,\nabla\Theta\rangle = 0 \tag{2.14}$$

and

$$(g''|\nabla\rho|^2 + g'\Delta\rho - g|\nabla\Theta|^2) + 2\frac{h'(g(\rho)^2)g}{h(g(\rho)^2)}\left(g'^2|\nabla\rho|^2 - g^2|\nabla\Theta|^2)\right) = 0. \tag{2.15}$$

Combining (2.10) and (2.14) it follows that

$$g'\Delta\rho = g\left(1 + 2\frac{h'(g^2(\rho))}{h(g^2(\rho))}g^2\right)|\nabla\Theta|^2. \tag{2.16}$$

From (2.9) we obtain

$$\Delta\rho = g(\rho)\cdot h(g^2(\rho))\left(1 + 2\frac{h'(g^2(\rho))}{h(g^2(\rho))}g^2(\rho)\right)|\nabla\Theta|^2. \tag{2.17}$$

Assume now that $g(\rho) \in [\varrho_0, \varrho_1]$.

If the Gauss curvature of the surface is negative, then according to (1.3),

$$\Delta\rho \geq \varrho_0\cdot h(\varrho_0^2)\left(1 + 2\frac{h'(\varrho_0^2)}{h(\varrho_0^2)}\varrho_0^2\right)|\nabla\Theta|^2. \tag{2.18}$$

If the Gauss curvature of the surface is positive, then according to (1.5), we have

$$\Delta\rho \geq \varrho_0\cdot h(\varrho_1^2)\left(1 + 2\frac{h'(\varrho_1^2)}{h(\varrho_1^2)}\varrho_1^2\right)|\nabla\Theta|^2. \tag{2.19}$$

We make use of following well-known proposition.

Proposition 2.2. *Let $u = \rho e^{i\Theta}$ be a C^1 surjection between the rings $A(r_1, r_2)$ and $A(s_1, s_2)$ of the complex plane. Then*

$$\int_{r_1\leq|z|\leq r_2}|\nabla\Theta|^2\,\mathrm{d}x\,\mathrm{d}y \geq 2\pi\log\frac{r_2}{r_1}. \tag{2.20}$$

For its proof see for example [10].

Theorem 2.3 (The main theorem)**.** *Let u be a ρ_Σ harmonic diffeomorphism between the Euclidean annular regions $A(r_1, 1)$ and annulus $\{z \in \mathbb{C} : \rho_0 < d_\Sigma(z, 0) < \rho_1\}$, of a Riemann surface Σ and let $\varrho_i = g(\rho_i)$, $i = 0, 1$. Then*

$$\frac{\rho_1}{\rho_0} \geq 1 + \frac{\varrho_0}{\rho_0}\log^2 r_1 \begin{cases} h(\varrho_0^2)\left(1/2 + \frac{h'(\varrho_0^2)}{h(\varrho_0^2)}\varrho_0^2\right), & \text{if } K \text{ is negative;} \\ h(\varrho_1^2)\left(1/2 + \frac{h'(\varrho_1^2)}{h(\varrho_1^2)}\varrho_1^2\right), & \text{if } K \text{ is positive.} \end{cases} \tag{2.21}$$

Remark 2.4. The Euclidean annulus in the domain is taken because of simplicity. But harmonicity does not depend on the metric of the domain, which means that we can take any other annulus which is conformally equivalent to this Euclidean annulus. Therefore, the statement of the theorem holds there as well, taking instead of the annulus $A(r_1, 1)$ an arbitrary annulus A of a Riemann surface (Σ, σ_0)

with the modulus $\mathrm{Mod}\,(A) = \frac{1}{2\pi}\log\frac{1}{r_1}$. Notice also that, the theorem does hold as well assuming that u is proper, which means that $\lim_{|z|\to 1} d_\Sigma(u(z),0) = \rho_1$ and $\lim_{|z|\to r_1} d_\Sigma(u(z),0) = \rho_0$. If K is negative, then $h'(\varrho_0^2)$ is positive and equation (2.21) makes sense for all ρ_0 and ρ_1. If K is positive, then $1/2 + \frac{h'(\varrho_1^2)}{h(\varrho_1^2)}\varrho_1^2$ must be positive, if we want to have a non-trivial inequality.

Proof. Let $\varphi_n : (\rho_0,\rho_1) \mapsto (\rho_0,\rho_1)$ be a sequence of non decreasing functions, constant in some small neighborhood of ρ_0, and satisfying the following conditions

$$0 \le \varphi_n'(s) \to 1 \text{ and } 0 \le \varphi_n''(s) \to 0 \text{ as } n \to \infty \tag{2.22}$$

for every $s \in (\rho_0,\rho_1)$. (See [10]) for an example of such sequence). Assume that u is a corresponding diffeomorphism. Let $\rho = |u|$ and let ρ_n be the function defined on $\{z : r_1 < |z| < 1\}$ by $\rho_n(z) = \varphi_n(\rho(z))$.

Then

$$\Delta\rho_n(z) = \varphi_n''(\rho(z))|\nabla\rho(z)|^2 + \varphi_n'(\rho(z))\Delta\rho(z).$$

By (2.22) it follows at once that

$$\Delta\rho_n(z) \to \Delta\rho(z) \text{ as } n \to \infty$$

for every $z \in A(r_1,1)$. Similarly we obtain

$$\frac{\partial\rho_n}{\partial r}(z) \to \frac{\partial\rho}{\partial r}(z) \text{ as } n \to \infty$$

uniformly on $\{z : |z| = r\}$ for every $r \in (r_1,1)$. By applying Green's formula for ρ_n on $\{z : r_1 + 1/n \le |z| \le r\}$, we obtain

$$\int_{|z|=r} \frac{\partial\rho_n}{\partial r}\,ds - \int_{|z|=r_1+1/n} \frac{\partial\rho_n}{\partial r}\,ds = \int_{r_1+1/n\le|z|\le r} \Delta\rho_n\,d\mu.$$

Since the function ρ_n is constant in some neighborhood of the circle $|z| = r_1 + 1/n$, it follows that

$$\int_{|z|=r} \frac{\partial\rho_n}{\partial r}\,ds = \int_{r_1+1/n\le|z|\le r} \Delta\rho_n\,d\mu.$$

Because of (2.17) and (2.22) it follows that the function $\Delta\rho_n$ is positive for every n. Hence, by applying Fatou's lemma, letting $n \to \infty$, we obtain

$$\int_{|z|=r} \frac{\partial\rho}{\partial r}\,ds \ge \int_{r_1\le|z|\le r} \Delta\rho\,d\mu.$$

Assume now that $K \le 0$. Next, by applying (2.18) and (2.20), we obtain

$$\begin{aligned}
\int_{|z|=r} \frac{\partial\rho}{\partial r}\,ds &\ge \int_{r_1\le|z|\le r} \Delta\rho\,d\mu \\
&= \int_{r_1\le|z|\le r} g(\rho)\cdot h(g^2(\rho))\left(1 + 2\frac{h'(g^2(\rho))}{h(g^2(\rho))}g^2(\rho)\right)|\nabla\Theta|^2 d\mu \\
&\ge g(\rho_0)\cdot h(g^2(\rho_0))\left(1 + 2\frac{h'(g^2(\rho_0))}{h(g^2(\rho_0))}g^2(\rho_0)\right)\int_{r_1\le|z|\le r} |\nabla\Theta|^2 d\mu \\
&\ge 2\pi g(\rho_0)\cdot h(g^2(\rho_0))\left(1 + 2\frac{h'(g^2(\rho_0))}{h(g^2(\rho_0))}g^2(\rho_0)\right)\log\frac{r}{r_1}.
\end{aligned}$$

It follows that

$$r\frac{\partial}{\partial r}\int_{|\zeta|=1}\rho\, ds(\zeta)\ge 2\pi g(\rho_0)\cdot h(g^2(\rho_0))\left(1+2\frac{h'(g^2(\rho_0))}{h(g^2(\rho_0))}g^2(\rho_0)\right)\log\frac{r}{r_1}.$$

Dividing by r and integrating over $[r_1,1]$ by r the previous inequality, we get

$$\int_{|\zeta|=1}\rho(\zeta)\, ds(\zeta)-\int_{|\zeta|=1}\rho(r_1\zeta)\, ds(\zeta)$$
$$\ge \pi g(\rho_0)\cdot h(g^2(\rho_0))\left(1+2\frac{h'(g^2(\rho_0))}{h(g^2(\rho_0))}g^2(\rho_0)\right)\log^2 r_1$$

i.e.

$$2\pi(\rho_1-\rho_0)\ge \pi g(\rho_0)\cdot h(g^2(\rho_0))\left(1+2\frac{h'(g^2(\rho_0))}{h(g^2(\rho_0))}g^2(\rho_0)\right)\log^2 r_1. \tag{2.23}$$

Thus (2.21) follows for this case. Similarly using (2.19) the case $K\ge 0$ can be established . □

3. Riemann case

Recall that the Riemann metric with the curvature 1 in the sphere S^2 is given by:

$$ds^2=h^2(|z|^2)=\frac{4|dz|^2}{(1+|z|^2)^2}.$$

It induces the following intrinsic distance function:

$$d_R(z,w)=2\arctan\left|\frac{z-w}{1+z\bar{w}}\right|. \tag{3.1}$$

The chordal distance is similar and is induced by stereograph projection:

$$d(z,w)=\frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}.$$

In both cases the isometries are

$$f(z)=e^{i\varphi}\frac{z-a}{1+\bar{a}z},\ a\in\mathbb{C},\varphi\in[0,2\pi).$$

They form a subgroup of the group $\mathrm{PGL}_2(\mathbb{C})$ of all Möbius transformations. It is denoted by PSU_2. This subgroup is isomorphic to the rotation group SO(3), which is the isometry group of the unit sphere in $\mathbb{R}^3$.

Assume now that u is harmonic in this setting. Equation (1.6) becomes

$$u_{z\bar{z}}-\frac{2\bar{u}}{1+|u|^2}u_z\cdot u_{\bar{z}}=0. \tag{3.2}$$

Notice this important example. The Gauss map of a surface Σ in $\mathbb{R}^3$ sends a point on the surface to the corresponding unit normal vector $\mathbf{n}\in\overline{\mathbb{C}}\cong S^2$. In terms of a conformal coordinate z on the surface, if the surface has *constant mean curvature*, its Gauss map $\mathbf{n}:\Sigma\mapsto\overline{\mathbb{C}}$, is a Riemann harmonic map [20] (see also [14] and [7] for a related topic).

If now we consider the geodesic polar coordinates (ρ,Θ) about the point $u(0)$ of the Riemann sphere S^2, then we have

$$ds^2=d\rho^2+\sin^2\rho d\Theta^2.$$

Let $f \in \mathrm{PSU}_2$ and take $w = f \circ u$.

Then, since PSU_2 is the group of isometries, according to Lemma 1.1 it follows that w is harmonic with respect to the Riemann metric.

Thus, if u is a harmonic mapping, then

$$w(z) = \frac{u(z) - u(0)}{1 + u(z)\overline{u(0)}} = g(\rho)e^{i\Theta}$$

is harmonic as well. Using (3.1) it follows that the inverse of the mapping $\varrho \mapsto d_R(\varrho, 0)$ is given by

$$g(\rho) = \tan\frac{\rho}{2}.$$

According to (2.17) we obtain

$$\Delta\rho = \frac{\sin 2\rho}{2}|\nabla\Theta|^2. \tag{3.3}$$

It follows that $\Delta\rho \geq 0$ if $\rho \in [0, \pi/2]$. This means that ρ is a subharmonic in the lower hemisphere. Moreover, from (3.3) it follows that,

$$\Delta\rho \geq \rho_0\frac{\sin 2\rho_1}{2\rho_1}|\nabla\Theta|^2, \text{whenever } \rho \in [\rho_0, \rho_1] \subset [0, \pi/2]. \tag{3.4}$$

The annulus $A(\tau, \sigma) = \{z : \tau \leq |z| \leq \sigma\}$ is conformally equivalent (and isometric with respect to Riemann metric) to the annulus

$$\{z \in \mathbb{C} : \rho_0 \leq d_R(z, w) \leq \rho_1\},$$

where

$$d_R(z, w) = 2\arctan\left|\frac{z - w}{1 + z\bar{w}}\right|,$$

and

$$\rho_0 = 2\arctan\tau, \quad \rho_1 = 2\arctan\sigma.$$

On the other hand if A_0 is an arbitrary doubly connected domain in $\mathbb{C}$ then it is conformally equivalent to an annulus $A(r_1, 1)$. If w is harmonic, a is analytic, and k is an isometry of S^2, i.e. if $k \in \mathrm{PSU}_2$, then $k \circ w \circ a$ is harmonic.

Having the previous fact in mind and following the same lines of the proof of Theorem 2.3, and using (3.4) instead of (2.19) the following theorem for Riemann harmonic mappings can be proved.

Theorem 3.1. *a) Let there be a Riemann harmonic diffeomorphism between the annular regions with the modulus* $\frac{1}{2\pi}\log\frac{1}{r_1}$ *and* $A_R(\rho_0, \rho_1, a) := \{w \in \overline{\mathbb{C}} : \rho_0 < d_R(w, a) < \rho_1\}$*, with* $\rho_1 < \pi/2$*. Then*

$$\frac{\rho_1}{\rho_0} \geq 1 + \frac{\sin 2\rho_1}{4\rho_1}\log^2 r_1. \tag{3.5}$$

b) In particular for $a = 0$*, we obtain that if there is a harmonic mapping between annular regions* $A(r_1, 1)$ *and* $A(\tau, \sigma)$ $(\sigma < 1)$*, of* $\mathbb{R}^2$*, then*

$$\frac{\arctan\sigma}{\arctan\tau} \geq 1 + \frac{\sigma(1 - \sigma^2)}{2(1 + \sigma^2)^2\arctan\sigma}\log^2 r_1. \tag{3.6}$$

Note that the condition $\rho_1 < \pi/2$*, (i.e.* $\sigma < 1$*) means that the annulus is contained in a hemisphere of* S^2*. The items a) and b) are equivalent.*

By applying Theorem 3.1 and having in mind the discussion after equation (3.2) we have.

Corollary 3.2. *Let Σ be a surface with constant mean curvature in $\mathbb{R}^3$. Let in addition $A_R(\rho_0, \rho_1, a)$ be an annulus in S^2 that lies in a hemisphere of S^2. Let $\mathbf{n}$ be the Gauss map of Σ, that maps a ring domain $A \subset \Sigma$ properly onto the annulus $A_R(\rho_0, \rho_1, a)$. Then*

$$\operatorname{Mod}(A) \leq \frac{1}{\pi}\sqrt{\frac{\rho_1 - \rho_0}{\rho_0} \cdot \frac{\rho_1}{\sin 2\rho_1}}, \tag{3.7}$$

where $\operatorname{Mod}(A)$ *is the conformal modulus of* A.

Notice that, in the case of minimal surfaces, the Gauss map is meromorphic. If the Gauss map is injective and meromorphic, then (3.7) is equivalent with (3.6), taking $r_1 = \frac{\tau}{\sigma}$, and this inequality can be proved directly. Notice also this interesting fact, the right hand side of inequality (3.7) does not depend on the surface Σ. For this topic see [19] and [4].

Example 3.3. Let $\Sigma = \{(u, v, w) : u^2 + v^2 = 1, w \in \mathbb{R}\}$ be a cylinder. Then Σ is a CMC surface. The conformal parametrization is given by $f(x, y) = (\cos x, \sin x, y)$, and the Gauss map is given by $\mathbf{n}(x, y) = (\cos x, \sin x)$. It is easy to see that $\mathbf{n}$ satisfies (3.2). Moreover, the image of the whole cylinder is the equator, and this means that the condition "$A_R(\rho_0, \rho_1, a)$ is an annulus in S^2 that lies in a hemisphere of S^2" in Corollary 3.2 is important.

In order to find examples of radial Riemann harmonic maps, we will put

$$w(z) = g(r)e^{i\varphi},$$

where g is a increasing or a decreasing function to be chosen. This will include all harmonic radial mappings.

Direct calculations yield

$$w_{z\bar{z}} = \frac{1}{4}\Delta w = \frac{1}{4r^2}\left(r^2 w_{rr} + r w_r + w_{\varphi\varphi}\right), \tag{3.8}$$

$$w_z w_{\bar{z}} = \frac{1}{4}\left(w_x^2 + w_y^2\right) = \frac{1}{4r^2}(r^2 w_r^2 - w_\varphi^2). \tag{3.9}$$

Inserting this to the harmonic equation, we obtain

$$r^2 g'' + r g' - g - \frac{2g}{1+g^2}(r^2 {g'}^2 - g^2) = 0.$$

Let $r = e^x$.

Setting $y(x) = g(e^x)$, we obtain

$$y'' - y = \frac{2y}{1+y^2}\left({y'}^2 - y^2\right),$$

where y is given by

$$\int \frac{1}{\sqrt{y^2 + c(1-y^2)^2}} dy = x + c_1. \tag{3.10}$$

Let $\sigma \leq 1$. Then by

$$r = \exp\left(\int_\sigma^y \frac{1}{\sqrt{z^2 + c(1-z^2)^2}} dz\right), \tag{3.11}$$

the inverse of the mapping $g$is given, satisfying the condition $r(\sigma) = 1$.

The integrand in previous integral is real for $\tau \le y \le \sigma$ if and only if

$$c \geq -(1/\tau + \tau)^{-2}. \tag{3.12}$$

Denote by $h_{\tau,\sigma}(y)$ the function

$$r = h_{\tau,\sigma}(y) = \exp \int_{\sigma}^{y} \frac{dx}{\sqrt{x^2 - (1/\tau + \tau)^{-2}(1 + x^2)^2}}.$$

It is extremal in the sense

$$h_{\tau,\sigma}(\tau) \le \exp\left(\int_{\sigma}^{\tau} \frac{1}{\sqrt{z^2 + c(1 + z^2)^2}} dz\right)$$

whenever condition (3.12) is satisfied.

As $h_{\tau,\sigma}(y)$ is an increasing function for $\tau \le y \le \sigma$, it follows that

$$w(z) = h^{-1}_{\tau,\sigma}(r)e^{i\varphi}$$

is a spherical harmonic mapping of the annulus $A(h_{\tau,\sigma}(\tau), 1)$ onto the annulus $A(\tau, \sigma)$.

Conjecture 1. *If $\sigma < 1$, and there exists a Riemann harmonic mapping of the annulus $A(r, 1)$ onto the annulus $A(\tau, \sigma)$, then*

$$r \geq h_{\tau,\sigma}(\tau). \tag{3.13}$$

This is similar to the Euclidean plane harmonic conjecture of J. C. C. Nitsche (see introduction of this paper).

Although the mapping

$$w(z) = h^{-1}_{\tau,1}(r)e^{i\varphi}$$

is a Riemann harmonic mapping of the annulus $A(h_{\tau,1}(\tau), 1)$ onto the annulus $A(\tau, 1)$ the case $\sigma = 1$ is excluded.

Namely for $\sigma < 1$

$$\lim_{\tau \to 1} h_{\tau,\sigma}(\tau) = 1,$$

but

$$\lim_{\tau \to 1} h_{\tau,1}(\tau) = e^{-\pi/2},$$

and this means that we can map, by means of Riemann harmonic diffeomorphisms, the annulus with modulus

$$\frac{1}{2\pi} \log\left(\frac{1}{e^{-\pi/2}}\right) = \frac{1}{4}$$

onto the annulus with arbitrarily small modulus.

This means that the condition $\sigma < 1$ is essential in Theorem 3.1.

The question arises, if is the constant $\frac{1}{4}$ the best (smallest) possible value of the modulus in this setting.

Figure 1 is shows the function $\frac{1}{2\pi} \log\left(\frac{1}{h_{\tau,1}(\tau)}\right)$.

For $\sigma < 1$,

$$\lim_{\tau \to \sigma - 0} h_{\tau,\sigma}(\tau) = 1,$$

and this verifies Theorem 3.1 in some sense.

In (3.6) it is shown that

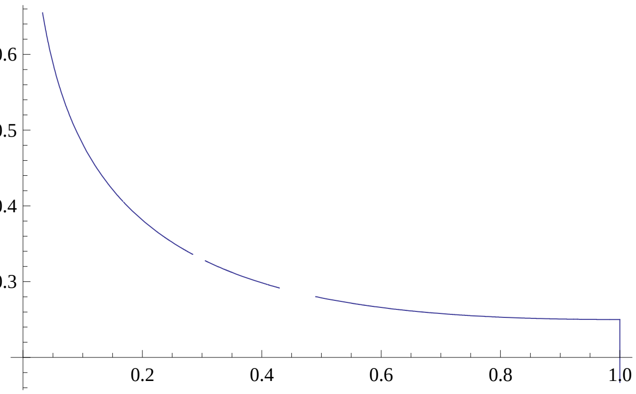

FIGURE 1. The modulus of extremal domains. The infimum is 1/4.

$$r \geq h_0(\sigma, \tau) := \exp\left(-\frac{2(1+\sigma^2)^2 \arctan\sigma(\arctan\sigma - \arctan\tau)}{\sigma(1-\sigma^2)\arctan\tau}\right). \tag{3.14}$$

Of course $h_{\tau,\sigma}(\tau) \geq h_0(\sigma,\tau)$. Figures 2 and 3 illustrate that our inequality is almost sharp when σ is not too close to 1.

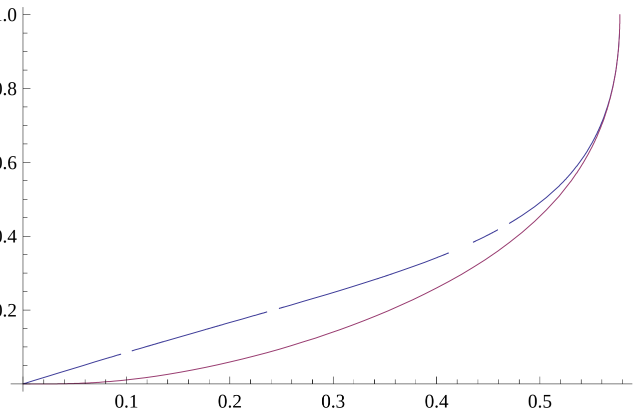

FIGURE 2. The case $\sigma = \sqrt{3}/3$, above is the function $h_{\tau,\sqrt{3}/3}(\tau)$ and below is the function $h_0(\sqrt{3}/3, \tau)$.

4. HYPERBOLIC CASE

This case has been studied in [24], but for the halfplane model. More precisely the following theorem has been proved in [24]: Let A_{ρ_1,ρ_2} be a round annulus in the hyperbolic plane H centered at u_0, that is, $A_{\rho_1,\rho_2} = \{u \in H : \rho_1 \leq \rho(u, u_0) \leq \rho_2\}$. Let $u\colon B \subset \mathbb{C} \to A_{\rho_1,\rho_2}$ be a harmonic diffeomorphism. Then the conformal modulus $\mathrm{Mod}(B) \leq C\sqrt{\rho_2 - \rho_1}\exp(-\rho_1)$, with some absolute constant C. According to Lemma 1.1 this corresponds to the inequality (4.3). We consider the unit disk

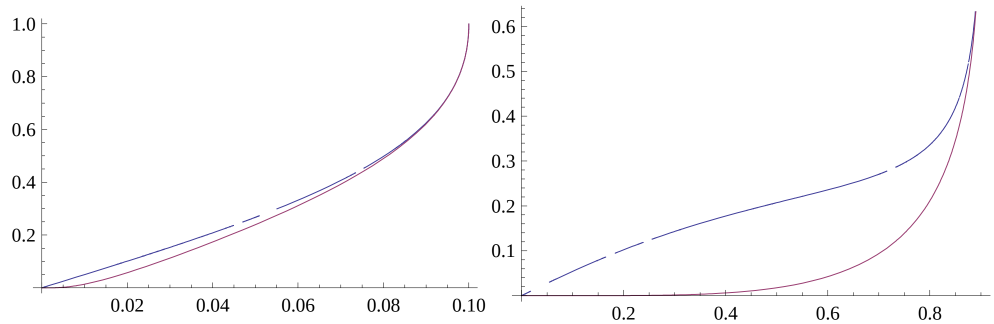

FIGURE 3. The left graphic corresponds to the case $\sigma = 1/10$, and the right one to the case $\sigma = 9/10$.

directly, in order to give an explicit inequality (see (4.4) below), and in order to analyze the sharpness of the result. We will consider a hyperbolic annulus as well. A similar argument can be repeated for the punctured unit disk. It can be considered as a special case of Theorem 2.3, however additional calculations are needed.

After that we will provide some examples of radial hyperbolic harmonic mappings of the hyperbolic unit disk, which show that inequality (4.4) is almost sharp. The author believes that, these examples are well known, but it seems that they haven't been considered for this setting.

If $u : \mathbb{U} \mapsto \mathbb{U}$ is a harmonic mapping with respect to the hyperbolic metric

$$\varrho ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}$$

then Euler-Lagrange equation of u is

$$u_{z\bar{z}} + \frac{2\bar{u}}{1-|u|^2} u_z \cdot u_{\bar{z}} = 0. \tag{4.1}$$

An important example of hyperbolic harmonic mapping is the Gauss map of a space-like surfaces with constant mean curvature H in the Minkowski 3-space $M^{2,1}$ (see [5], [17] and [25]).

As in the Riemann case we consider the geodesic polar coordinates (ρ, Θ) about the point $u_0 = u(0)$ of the unit disc $\mathbb{U}$. We have

$$\rho = \log\left(\frac{1+|\frac{u-u_0}{1-u\overline{u_0}}|}{1-|\frac{u-u_0}{1-u\overline{u_0}}|}\right)$$

and consequently

$$\tanh\frac{\rho}{2} = |\frac{u-u_0}{1-u\overline{u_0}}|.$$

Hence

$$\frac{u-u_0}{1-u\overline{u_0}} = \tanh\frac{\rho}{2}e^{i\Theta}.$$

Setting

$$w = \frac{u-u_0}{1-u\overline{u_0}}$$

we obtain that

$$u = \frac{w+u_0}{1+w\overline{u_0}}.$$

Since the mappings $\{w = e^{i\varphi}\frac{z-a}{1-z\bar{a}}, |a|<1\}$ are isometries of the hyperbolic unit disk, according to Lemma 1.1 the mapping w is harmonic with respect to the hyperbolic metric.

In this case $\varrho = g(\rho) = \tanh(\frac{\rho}{2})$.

According to (2.17) and (2.14) it follows that

$$\Delta\Theta + (2/\sinh\rho + 2)\langle\nabla\rho, \nabla\Theta\rangle = 0,$$

and

$$g'\Delta\rho = 2\frac{g(\rho(z))}{1-g^2}g^2|\nabla\Theta|^2 + g|\nabla\Theta|^2.$$

Thus

$$\Delta\rho = \frac{\sinh 2\rho}{2}|\nabla\Theta|^2.$$

Theorem 2.3 yields that

$$\frac{\rho_1}{\rho_0} \geq 1 + \frac{\varrho_0}{\rho_0}\log^2 r_1 h(\varrho_0^2)\left(1/2 + \frac{h'(\varrho_0^2)}{h(\varrho_0^2)}\varrho_0^2\right) = 1 + \frac{\varrho_0}{\rho_0}\frac{1+\varrho_0^2}{(1-\varrho_0^2)^2}\log^2 r_1. \tag{4.2}$$

Therefore

$$\frac{\rho_1}{\rho_0} \geq 1 + \frac{\sinh 2\rho_0}{2\rho_0}\log^2 r_1, \tag{4.3}$$

or equivalently

$$\log\left(\frac{1+\varrho_1}{1-\varrho_1} : \frac{1+\varrho_0}{1-\varrho_0}\right) \geq \varrho_0\frac{1+\varrho_0^2}{(1-\varrho_0^2)^2}\log^2 r_1. \tag{4.4}$$

Hence, if w is a hyperbolic harmonic mapping between the annuli $A(r_1, 1)$ and $A(\varrho_0, \varrho_1)$, then inequality (4.4) holds.

Now as in the Riemann case, if A_0 is any doubly connected domain in $\mathbb{C}$ with the modulus $\frac{1}{2\pi}\log\frac{1}{r_1}$ and A_1 is any annulus in the hyperbolic disk that is isometric to an annulus $A(\varrho_0, \varrho_1)$, then (4.4) holds.

The following theorem establishes a corresponding inequality for hyperbolic annulus $A(1/R, R)$. See Subsection 1.1.1.

Theorem 4.1. *Let there be a λ_R harmonic diffeomorphism between the annular region $A(r_1, 1)$ of the Euclidean plane and $\{z \in \mathbb{C} : 0 < \rho_0 < \int_1^{|z|} h_R(t^2)dt < \rho_1\}$, of the hyperbolic annulus $A(1/R, R)$ and let g be the inverse of the function $\omega(r) = \int_1^r h_R(t^2)dt$. Let in addition $\varrho_i = g(\rho_i)$, $i = 0, 1$. Then*

$$\rho_1 - \rho_0 \geq \varrho_0 \sec\frac{\pi\log\varrho_0}{2\log R}\left(\log R - 2\varrho_0\log R + \pi\varrho_0\tan\frac{\pi\log\varrho_0}{2\log R}\right)\frac{\pi\log^2 r_1}{4\log^2 R}. \tag{4.5}$$

Proof. The only part which is different from the proof of Theorem 2.3 is the fact that

$$\rho = \int_1^{g(\rho)} h_R(t^2)dt. \tag{4.6}$$

Now as in (2.23)

$$\rho_1 - \rho_0 \geq \varrho_0\left(h_R(\varrho_0^2)/2 + h'_R(\varrho_0^2)\varrho_0^2\right)\log^2 r_1. \tag{4.7}$$

The function ω is given by

$$\omega(r) = 2\text{arctanh}\left(\tan\frac{\pi\log r}{4\log R}\right).$$

The rest of the proof is a standard calculation. □

It follows from (4.5) that,

$$r_1 \geq r(\rho_0, \rho_1) > 0,$$

however, if we fix ρ_0 then

$$\lim_{\rho_1 \to \infty} r(\rho_0, \rho_1) = 0.$$

The previous fact, implies that inequality (4.5) has a local character. It seems that such a global estimate does not exist.

Remark 4.2. It follows from Theorem 4.1 that: If $\mathbb{U}^* = \mathbb{U} \setminus \{0\}$ is the punctured unit disc with hyperbolic metric, and $A(1/R, R)$ is an annulus with hyperbolic metric, then there is no surjective λ_R harmonic diffeomorphism of $\mathbb{U}^*$ onto $A(\tau, \sigma)$, with $1/R < \tau < \sigma < R$. The question arises, if there exists a λ_R harmonic diffeomorphism of $\mathbb{U}^*$ onto $A(1/R, R)$. For this problem we refer the interested reader to [16], [6].

As in the Riemann case, we are going to find examples of radial hyperbolic harmonic maps. We put

$$w(z) = g(r)e^{i\varphi},$$

where g is a increasing function to be chosen. This will include all harmonic radial mappings. These examples, will show that (4.4) is almost sharp and will suggest a J. C. C. Nitsche type conjecture for hyperbolic harmonic mappings.

Inserting (3.8) and (3.9) into the hyperbolic harmonic equation (4.1), we obtain

$$r^2 g'' + r g' - g + \frac{2g}{1-g^2}(r^2 {g'}^2 - g^2) = 0.$$

Setting $y(x) = g(e^x)$, i.e. $r = e^x$, we obtain

$$y'' - y = \frac{2y}{y^2 - 1}\left({y'}^2 - y^2\right),$$

where y is given by

$$\int \frac{1}{\sqrt{y^2 + c(1-y^2)^2}} dy = x + c_1. \tag{4.8}$$

Let $\sigma \leq 1$. Then by

$$r = q_{c,\sigma}(y) := \exp\left(\int_\sigma^y \frac{1}{\sqrt{z^2 + c(1-z^2)^2}} dz\right), \tag{4.9}$$

is given the inverse of the mapping g, satisfying the condition $q_{c,\sigma}(\sigma) = 1$.

The integrand in previous integral is real for $\tau \leq y \leq \sigma$ if and only if

$$c \geq -(1/\tau - \tau)^{-2}.$$

Define the function $p_{\tau,\sigma}(y)$ by

$$r = p_{\tau,\sigma}(y) = \exp \int_\sigma^y \frac{dz}{\sqrt{z^2 - (1/\tau - \tau)^{-2}(1-z^2)^2}}.$$

It is extremal in the following sense

$$p_{\tau,\sigma}(\tau) \leq q_{c,\sigma}(\tau)$$

for every

$$c \geq -(1/\tau - \tau)^{-2}.$$

As $p_{\tau,\sigma}(y)$ is an increasing function for $\tau \leq y \leq \sigma$, it follows that

$$w(z) = p^{-1}_{\tau,\sigma}(r)e^{i\varphi}$$

is a hyperbolic harmonic diffeomorphism of the annulus $A(p_{\tau,\sigma}(\tau), 1)$ onto the annulus $A(\tau, \sigma)$. Similarly, we can construct harmonic diffeomorphism with decreasing $g(r)$.

Notice that $w(z) = q^{-1}_{c,1}(r)e^{i\varphi}$, with $c = 0$, is the identity, and for $c > 0$ is a hyperbolic harmonic diffeomorphism between the annulus $A(r_{c,1}(0), 1)$ and the punctured unit disc $A(0, 1)$.

For every $\sigma \leq 1$,

$$\lim_{\tau \to \sigma - 0} p_{\tau,\sigma}(\tau) = 1.$$

Conjecture 2. *If $\sigma \leq 1$, and there exists a hyperbolic harmonic mapping of the annulus $A(r, 1)$ onto the annulus $A(\tau, \sigma)$, then $r \geq p_{\tau,\sigma}(\tau)$.*

Let $s \in [0, 1/2]$ and take $\tau = 1 - 2s$ and $\sigma = 1 - s$. Then our example states that

$$r \geq f_1(s) = p_{1-2s,1-s}(1-2s) = \exp \int_{1-s}^{1-2s} \frac{dx}{\sqrt{x^2 - (1/(1-2s) - 1 + s)^{-2}(1 - x^2)^2}}.$$

Inequality (4.2) asserts that

$$r \geq f_2(2) = \exp\left(-\sqrt{\frac{(1 - (1-2s)^2)^2 \log \frac{2-s}{1-s}}{(1 + (1-2s)^2)(1-2s)}}\right).$$

It can be verified that if $s \in [0, 1/2]$ then $f_1(s) \geq f_2(s)$. See figure 4, which shows that inequality (4.4) is almost sharp for these cases.

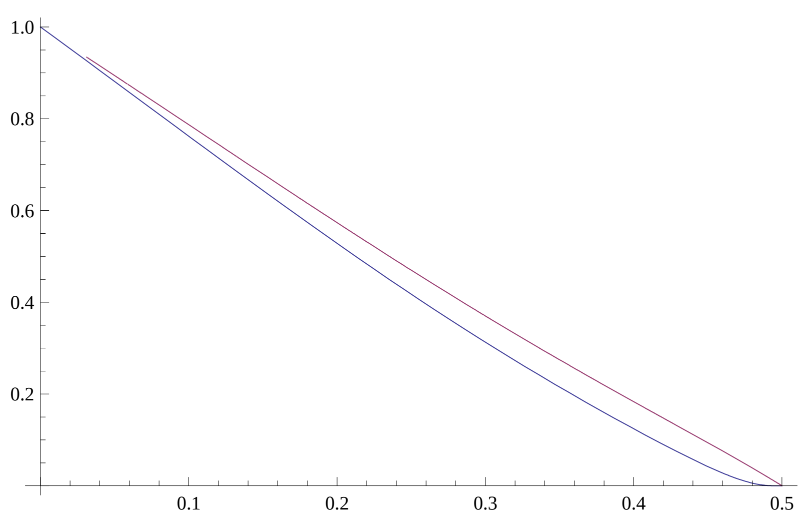

FIGURE 4. The function bellow is f_2 and above is f_1.

Remark 4.3. Since, the Gauss map of a spacelike constant mean curvature hypersurface of Minkowski space is a hyperbolic harmonic mapping ([5]), it seems that, a similar statement which corresponds to Corollary 3.2 holds in hyperbolic case as well.

Acknowledgment. The author thanks the referee for his helpful suggestions concerning the presentation of this paper.

References

[1] Ahlfors, L.; Sario L. *Riemann Surfaces (1st ed.),* Princeton, New Jersey: Princeton University Press, p. 204 (1960). 1.1.1

[2] Astala, K.; Iwaniec, T.; Martin, G. J.; Onninen, J. *Extremal mappings of finite distortion.* Proc. London Math. Soc. (3) 91 (2005), no. 3,655–702. 1

[3] Bshouty, D.; Hengartner, W; *Univalent harmonic mappings in the plane*, Ann. Univ. Mariae Curie-Sklodowska Sect. A XLVIII (1994). 1

[4] Bshouty, D,; Weitsman, A. *On the Gauss map of minimal graphs.* Complex Var. Theory Appl. 48 (2003), no. 4, 339–346. 3

[5] Choi, H.; Treibergs, A. *Gauss maps of spacelike constant mean curvature hypersurfaces of Minkowski space,* J. Differential Geometry 32 (1990), 775-817. 4, 4.3

[6] Collin, P.; Rosenberg, H. *Construction of harmonic diffeomorphisms and minimal graphs*, arXiv:math/0701547v1. 4.2

[7] Dorfmeister, J.; Pedit, F.; Wu, H. *Weierstrass type representation of harmonic maps into symmetric spaces.* Comm. Anal. Geom. 6 (1998), no. 4, 633–668. 3

[8] Iwaniec, T; L. Kovalev; Onninen, J.: *Harmonic mappings of an annulus, Nitsche conjecture and its generalizations* arXiv:0903.2665 (March 2009). 1

[9] Kalaj, D. *On the univalent solution of PDE* $\Delta u = f$ *between spherical annuli.* J. Math. Anal. Appl. 327 (2007), no. 1, 1–11. 1

[10] ——— *On the Nitsche conjecture for harmonic mappings in* $\mathbb{R}^2$ *and* $\mathbb{R}^3$. Israel J. Math. 150 (2005), 241–251. 1, 2, 2

[11] ——— *On the Nitsche's conjecture for harmonic mappings.* Math. Montisnigri 14 (2001), 89–94. 1

[12] Kalaj, D.; Mateljević, M. *Inner estimate and quasiconformal harmonic maps between smooth domains*, J. Anal. Math. **100** (2006), 117–132. 1.1.3

[13] Lehto O.; Virtanen, K.I. *Quasiconformal mapping*, Springer-verlag, Berlin and New York, 1973. 1

[14] Lerner, D; Sterling, I. *Holomorphic potentials, symplectic integrators and CMC surfaces.* (English summary) Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994), 7390, A K Peters, Wellesley, MA, 1996. 3

[15] Lyzzaik, A. *The modulus of the image of annuli under univalent harmonic mappings and a conjecture of J. C. C. Nitsche.* J. London Math. soc., (2) 64 (2001), pp. 369-384. 1

[16] Marković, V. *Harmonic diffeomorphisms and conformal distortion of Riemann surfaces*; Comm. Anal. Geom. 10 (2002), p.847–876. 4.2

[17] Milnor, T. *Harmonic maps and classical surface theory in Minkowski* 3*-space.* Trans. Amer. Math. Soc. 280 (1983), no. 1, 161–185. 4

[18] Nitsche, J.C.C. *On the modulus of doubly connected regions under harmonic mappings*, Amer. Math. Monthly 69 (1962), 781-782. 1

[19] Ros, A. *The Gauss map of minimal surfaces.* Differential geometry, Valencia, 2001, 235–252, World Sci. Publ., River Edge, NJ, 2002. 3

[20] Ruh, E.; Vilms, J. *The tension field of the Gauss map.* Trans. Amer. Math. Soc. 149 1970 569–573. 3

[21] G. Schober, *Planar harmonic mappings, Computational Methods and Function Theory Proceedings*, Valparaiso, 1989, Lecture Notes in Mathematics 1435 (Springer, 1990) 171-176. 1

[22] Schoen, R.; Yau, S. T. *Lectures on harmonic maps.* Conference Proceedings and Lecture Notes in Geometry and Topology, II. International Press, Cambridge, MA, 1997. vi+394 pp. 1.1.3

[23] Strebel, K. *Quadratic differentials*, Springer-Verlag, Berlin Heidelberg, New York, Tokyo 1984. 2

[24] Han, Zheng-Chao, *Remarks on the geometric behavior of harmonic maps between surfaces. Chow, Ben (ed.) et al., Elliptic and parabolic methods in geometry.* Proceedings of a workshop, Minneapolis, MN, USA, May 23–27, 1994. Wellesley, MA: A K Peters. 57-66 (1996). 1, 4

[25] Wan, T. *Constant mean curvature surface, harmonic maps, and universal Teichmller space.* J. Differential Geom. 35 (1992), no. 3, 643–657. 4

[26] Weitsman, A. *Univalent harmonic mappings of Annuli and a conjecture of J.C.C. Nitsche,* Israel J. Math, 124(2001). 1

University of Montenegro, Faculty of Natural Sciences and Mathematics, Cetinjski put b.b. 81000 Podgorica, Montenegro

E-mail address: davidk@t-com.me