ON THE QUASICONFORMAL SELF-MAPPINGS OF THE UNIT BALL SATISFYING THE PDE $\Delta u = g$

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Abstract. It is proved that the family of K quasiconformal mappings of the unit ball onto itself satisfying PDE $\Delta u = g$, $g \in C(\overline{B^n})$, $u(0) = 0$ is a uniformly Lipschitz family. In addition is proved that, the Lipschitz constant tends to 1 as $K \to 1$ and $|g|_\infty \to 0$. This generalizes a similar two-dimensional case treated in [11] and solved the problem started in [15]. According to Fefferman's theorem, every analytic bi-holomorphic mapping between two smooth domains has C^1 extension to the boundary, and therefore the class of bi-holomorphic mappings between smooth domains, is contained in the class of harmonic quasiconformal mappings ($\Delta u = 0$). Therefore our results can be considered as extensions of Fefferman's theorem.

Contents

1. Introduction and statement of the main results

A twice differentiable function u defined in an open subset Ω of Euclidean space R^n will be called *harmonic* if it satisfies the differential equation

$$\Delta u(x) := D_{11}u(x) + D_{22}u(x) + \cdots + D_{nn}u(x) = 0.$$

In this paper B^n denotes the unit ball in $\mathbb{R}^n$, and S^{n-1} denotes the unit sphere. Also we will assume that $n > 2$ (the case $n = 2$ has been already treated in [11]). We will consider vector norm $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$ and two matrix norms: trace norm $|A|_2 := (\text{trace } AA^t)^{1/2} = (\sum_{i,j=1}^n a_{i,j}^2)^{1/2}$ and induced norm $|A| = \sup\{|Ax| : |x| = 1\}$.

A homeomorphism $u : \Omega \to \Omega'$ between two open subsets Ω and Ω' of Euclid space R^n will be called a K ($K \geq 1$) *quasi-conformal* or shortly a q.c mapping if

(i) u is absolutely continuous function in almost every segment parallel to some of the coordinate axes and there exist the partial derivatives which are locally L^n integrable functions on Ω. We will write $u \in ACL^n$ and

1991 *Mathematics Subject Classification.* Primary 30C65; Secondary 31B05.
Key words and phrases. Quasiconformal maps, PDE, Lipschitz condition.

(ii) u satisfies the condition

$$|\nabla u(x)|^n/K \le J_u(x) \le Kl(\nabla u(x))^n,$$

at almost everywhere x in Ω where

$$l(\nabla u(x)) := \inf\{|\nabla u(x)\zeta| : |\zeta| = 1\}$$

and $J_u(x)$ is the Jacobian determinant of u (see [28]).

Notice that, for a continuous mapping u the condition (i) is equivalent with that u belongs to the Sobolev space $W^1_{n,\mathrm{loc}}(\Omega)$.

Let P be Poisson kernel i.e. the function

$$P(x,\eta) = \frac{1-|x|^2}{|x-\eta|^n},$$

and let G be Green function i.e. the function

$$G(x,y) = c_n\left(\frac{1}{|x-y|^{n-2}} - \frac{1}{(|\,x|y| - y/|y|\,|)^{n-2}}\right) \tag{1.1}$$

where $c_n = \frac{1}{(n-2)\Omega_{n-1}}$, and Ω_{n-1} is the measure of S^{n-1}. Both P and G are harmonic for $|x| < 1$, $x \ne y$.

Let $f : S^{n-1} \to \mathbb{R}^n$ be a bounded integrable function on the unit sphere S^{n-1} and let $g : B^n \mapsto \mathbb{R}^n$ be continuous. The solution of the equation (in the sense of distributions) $\Delta u = g$ in the unit ball satisfying the boundary condition $u|_{S^{n-1}} = f \in L^1(S^{n-1})$ is given by

$$u(x) = P[f](x) - G[g](x) := \int_{S^{n-1}} P(x,\eta)f(\eta)d\sigma(\eta) - \int_{B^n} G(x,y)g(y)dy, \tag{1.2}$$

$|x| < 1$. Here $d\sigma$ is Lebesgue $n-1$ dimensional measure of Euclid sphere satisfying the condition: $P[1](x) \equiv 1$. It is well known that if f and g are continuous in S^{n-1} and in $\overline{B^n}$ respectively, then the mapping $u = P[f] - G[g]$ has a continuous extension $\tilde{u}$ to the boundary and $\tilde{u} = f$ on S^{n-1}.

We will consider those solutions of the PDE $\Delta u = g$ that are quasiconformal as well.

It seems that the family of q.c. harmonic mappings has first been considered in ([20]). Recent papers [12]–[14] and [23] bring much light on the topic of quasiconformal harmonic mappings on the plane.

In this paper we continue to study the same problem in the space. It was started in the paper [15]. The problem in the space is much more complicated because of lack of the techniques of complex analysis.

The following theorem gives a positive answer to the conjecture raised by the author in ([15]): that a q.c. harmonic self-mapping of the unit ball is Lipschitz continuous with Lipschitz constant depending only on a quasiconformality constant K. It is a generalization of an analogous theorem for the unit disk due to author and Pavlović ([11]). See also [15]) and [26]. It is the main result of the paper.

Theorem 1.1. *Let $K \ge 1$ be arbitrary, let $n \in \mathbb{N}$ and let $g \in C(\overline{B^n})$. Then there exist constants $M_1'(n,K)$ and $M_2'(n,K)$ such that: if u is K quasiconformal selfmapping of the unit ball B^n satisfying the PDE $\Delta u = g$, with $u(0) = 0$ then:*

$$|u(x) - u(y)| \le (M_1'(K,n) + M_2'(K,n)|g|_\infty)|x-y|,\ \ x,y \in B^n. \tag{1.3}$$

Moreover $M_1'(n,K) \to 1$ as $K \to 1$.

It is important to notice that, the class of harmonic functions (mappings) contains itself the class holomorphic functions (mappings). Therefore the class of harmonic automorphisms of the unit ball is a subclass of harmonic self-mappings of the unit ball. Having in mind the fact that, they have an extension across the boundary the unit ball, it follows that they are quasi-conformal. On the other hand according to Fefferman's theorem ([6]), every analytic bi-holomorphic mapping between two smooth domains has C^1 extension to the boundary, and therefore the class of bi-holomorphic mappings between smooth domains, is contained in the class of harmonic quasiconformal mappings. Therefore our results can be considered as extensions of Fefferman's theorem.

The proof of Theorem 1.1, given in Sections 3, depends on the following proposition:

Proposition 1.2. [13] *Let $u : B^n \to \Omega$ be twice differentiable q.c. mapping of the unit ball onto the bounded domain Ω with C^2 boundary satisfying the differential inequality:*

$$|\Delta u| \leq A|\nabla u|^2 + B,\ \ A, B \geq 0.$$

Then ∇u is bounded and u is Lipschitz continuous.

One of the advantages of Theorem 1.1 in relation to Proposition 1.2 is that, in Theorem 1.1 the Lipschitz constant do not depend on the mapping u, contrary the statement of Proposition 1.2.

It also depends on Mori's theorem in the theory of quasiconformal mappings:

Proposition 1.3. [5] *If u is a K quasi-conformal self-mapping of the unit ball B^n with $u(0) = 0$, then there exists a constant $M_1(n, K)$, satisfying the condition $M_1(n, K) \to 1$ as $K \to 1$, such that*

$$|u(x) - u(y)| \leq M_1(n, K)|x - y|^{K^{1/(1-n)}}. \tag{1.4}$$

See also [2] *for some constant that is not asymptotically sharp.*

The mapping $|x|^{-1+K^{1/(1-n)}}x$ shows that the exponent $K^{1/(1-n)}$ is optimal in the class of arbitrary $K-$ quasiconformal homeomorphisms.

2. Auxiliary results

By S and T we denote the spherical coordinates:

$$S : K_0^n = [0, 1] \times [0, \pi] \times \cdots \times [0, \pi] \times [0, 2\pi] \mapsto B^n$$

and

$$T : K^{n-1} = [0, \pi] \times \cdots \times [0, \pi] \times [0, 2\pi] \mapsto S^{n-1}$$

$(S(r, \theta_0, \dots, \theta_{n-2}, \varphi) = rT(\theta_1, \dots, \theta_{n-2}, \varphi))$, defined by $S = (x_1, x_2, ..., x_{n-1})$,

$$\begin{array}{rcl} x_1 &=& r\cos\theta_1, \\ x_2 &=& r\sin\theta_1\sin\theta_2, \\ &\vdots& \\ x_n &=& r\sin\theta_1\sin\theta_2\dots\sin\theta_{n-2}\cos\varphi, \\ x_{n+1} &=& r\sin\theta_1\sin\theta_2\dots\sin\theta_{n-2}\sin\varphi. \end{array}$$

Then we have:

$$\det S'(r, \theta_1, \dots, \theta_{n-2}, \varphi) = r^{n-1}\sin^{n-2}\theta_1\cdots\sin\theta_{n-2}. \tag{2.1}$$

We will use the notations $\theta = (\theta_1, \dots, \theta_{n-2}, \varphi)$ and $\theta_{n-1} = \varphi$.

Lemma 2.1. *Let u be harmonic function defined on the unit ball and assume that its derivative $v = \nabla u$ is bounded on the unit ball (or equivalently, let u be Lipschitz continuous). Then there exists a mapping $A \in L^\infty(S^{n-1})$ defined on the unit sphere S^{n-1} such that $\nabla u(x) = P[A](x)$ and for almost every $\eta \in S^{n-1}$ there holds the relation*

$$\lim_{r\to 1-} \nabla u(r\eta) = A(\eta). \tag{2.2}$$

Moreover the function $f \circ T$ is differentiable almost everywhere in K^{n-1} and there holds

$$A(T(\theta)) \cdot T'(\theta) = (f \circ T)'(\theta).$$

Proof. For the prof of the first statement of the lemma see for example [1, Theorem 6.13 and Theorem 6.39].

Next, since

$$\begin{aligned}|\frac{\partial}{\partial\theta_i} u(S(r,\theta))| = |r\nabla u(S(r,\theta))\frac{\partial}{\partial\theta_i}T(\theta)| &\le |r\nabla u(S(r,\theta))| \cdot |\frac{\partial}{\partial\theta_i}T(\theta)| \\ &\le \mathrm{essup}_\theta |A(T(\theta))| \cdot |\frac{\partial}{\partial\theta_i}T(\theta)|,\end{aligned}$$

the Lebesgue Dominated Convergence Theorem yields

$$\begin{aligned} f(T(\theta)) &= \lim_{r\to 1-} u(S(r,\theta)) \\ &= \lim_{r\to 1-} \int_{\theta_i^0}^{\theta_i} \frac{\partial}{\partial\theta_i} u(S(r,\theta))d\theta_i + f(T(\theta^0)) \\ &= \int_{\theta_i^0}^{\theta_i} \lim_{r\to 1-} \frac{\partial}{\partial\theta_i} u(rS(\theta))d\theta_i + f(T(\theta^0)) \\ &= \int_{\theta_i^0}^{\theta_i} \lim_{r\to 1-} r\nabla u(S(r,\theta))\frac{\partial}{\partial\theta_i}T(\theta)d\theta_i + f(T(\theta^0)) \\ &= \int_{\theta_i^0}^{\theta_i} A(T(\theta)) \cdot \frac{\partial}{\partial\theta_i}T(\theta)d\theta_i + f(T(\theta^0)). \end{aligned} \tag{2.3}$$

Differentiating in θ_i we get for every $i \in \{1, \dots, n-1\}$

$$\frac{\partial}{\partial\theta_i} f(T(\theta)) = A(T(\theta)) \cdot \frac{\partial}{\partial\theta_i}T(\theta)$$

almost everywhere in K^{n-1}.

Hence we have:

$$A(T(\theta)) \cdot T'(\theta) = (f \circ T)'(\theta)$$

almost everywhere in S^{n-1}.

□

Lemma 2.2. *Let u be a harmonic Lipschitz continuous mapping defined in the unit ball B^n. Denote by ∇u its extension up to the boundary $S^{n-1} = \partial B^n$, which exists almost everywhere in S^{n-1}. Then for $x \in B^n$*

$$|\nabla u(x)| \le \text{ ess sup}_{|\eta|=1} |\nabla u(\eta)|,$$

where $|\cdot|$ is trace norm or induced norm.

Proof. Let $u = (u_1, \dots u_n)$. For all (i,j) the function $u_{i,j} = \frac{\partial u_i}{\partial x_j}$ is bounded and harmonic. Hence there exists a bounded integrable function $g_{i,j}$ defined on the unit sphere such that $u_{i,j} = P[g_{i,j}]$. In other words

$$\nabla u(x) = \int_{S^{n-1}} g(\eta) P(x,\eta) d\sigma(\eta)$$

where $g(\eta)$ is $n \times n$ dimensional matrix $(g_{i,j}(\eta))_{i,j=1}^n$ and it coincides with $\nabla u(\eta)$. By definition, for the trace norm we have

$$\begin{aligned}|\nabla u(x)|_2^2 &= \text{trace } \nabla u(x) \nabla u(x)^t \\ &= \text{trace} \int_{S^{n-1}} g(\eta)P(x,\eta)d\sigma(\eta) \left(\int_{S^{n-1}} g(\eta)P(x,\eta)d\sigma(\eta)\right)^t \\ &\leq \int_{S^{n-1}} \text{trace } g(\eta)g(\eta)^t P(x,\eta)d\sigma(\eta).\end{aligned}$$

Hence

$$\begin{aligned}|\nabla u(x)|_2^2 &\leq \text{ess sup}\{\text{trace } g(\eta)g(\eta)^t : \eta \in S^{n-1}\} \int_{S^{n-1}} P(x,\eta)d\sigma(\eta) \\ &= \text{ess sup}\{\text{trace } g(\eta)g(\eta)^t\}.\end{aligned}$$

Thus we obtain

$$|\nabla u(x)|_2 \leq \text{ess sup}_{\eta \in S^{n-1}} |\nabla u(\eta)|_2.$$

For the induced norm we have

$$|A| = \max\{\langle Ah, k\rangle : |h| = |k| = 1\}.$$

Thus for $|h| = |k| = 1$ we have

$$\begin{aligned}\langle \nabla u(x)h, k\rangle &= \int_{S^{n-1}} \langle g(\eta)h, k\rangle\, P(x,\eta)d\sigma(\eta) \\ &\leq \int_{S^{n-1}} |g(\eta)| P(x,\eta)d\sigma(\eta) \\ &\leq \text{ess sup}_{|\eta|=1} |g(\eta)| \int_{S^{n-1}} P(x,\eta)d\sigma(\eta).\end{aligned}$$

The proof is completed. □

Lemma 2.3. *For every $\alpha < n$ the potential type integral*

$$I(x) = \int_{B^n} \frac{dy}{|x-y|^\alpha}$$

exists for every $x \in \mathbb{R}^n$, and achieves its maximum for $x = 0$. Furthermore:

$$I(0) = \frac{1}{n-\alpha}\Omega_{n-1}. \tag{2.4}$$

If $|x| = 1$ and $\alpha = n-1$, then

$$I(x) = \frac{2\Gamma(\frac{n}{2})}{(n-1)\sqrt{\pi}\Gamma(\frac{n-1}{2})}\Omega_{n-1}. \tag{2.5}$$

Moreover there exists a decreasing function ϕ defined on $[0, +\infty)$ such that $I(x) = \phi(r)$ on the sphere $S^{n-1}(0, r)$, $r > 0$.

Proof. Let $A = B^n \setminus B^n(x,1)$ and $B = B^n \cap B^n(x,1)$. Then $B^n = A \cup B$. If $y \in A$ then $|y-x| \geq |y|$. Thus

$$\int_A \frac{dy}{|x-y|^\alpha} \leq \int_A \frac{dy}{|y|^\alpha}.$$

On the other hand $B = -B + x$. Thus

$$\int_B \frac{dy}{|x-y|^\alpha} = \int_B \frac{dy}{|y|^\alpha}.$$

Hence

$$I(x) = \int_{B^n} \frac{dy}{|x-y|^\alpha} \leq I(0) = \int_{B^n} \frac{dy}{|y|^\alpha}.$$

Introducing the spherical coordinates centered on 0 and on the point x on the integrals $I(0)$ and $I(x)$, respectively we obtain the relations (2.4) and (2.5).

Using the similar argument it follows that ϕ is decreasing.

Lemma 2.4. [15]. *The integral*

$$I = \int_{S^{n-1}} |a-\eta|^\gamma d\sigma(\eta),$$

$a \in S^{n-1}$ *converges if and only if* $\gamma > 1-n$. *If* $\gamma = 2-n$ *then* $I = 1$.

□

Lemma 2.5. *Let* ρ *be a bounded (absolutely) integrable function defined on a bounded domain* $\Omega \subset \mathbb{R}^n$. *Then the potential type integral*

$$I(x) = \int_\Omega \frac{\rho(y)dy}{|x-y|^\alpha}$$

belongs to the space $C^p(\mathbb{R}^n)$, $p \in \mathbb{N}$ *such that* $\alpha + p < n$. *Moreover*

$$\nabla I(x) = \int_\Omega \nabla \frac{1}{|x-y|^\alpha} \rho(y)dy.$$

For the proof see for example [24, p. 24-26].

Lemma 2.6. *If* g *is continuous on* $\overline{B}^n$, *then the mapping* $G[g]$ *has a bounded derivative i.e. it is Lipschitz continuous. Moreover* $\nabla G[g]$ *has a continuous extension to the boundary and there holds*

$$\nabla G[g](\eta)h = \int_{B^n} \frac{\langle \eta, h\rangle}{\Omega_{n-1}} \frac{1-|y|^2}{|\eta-y|^n} g(y)dy,$$

for $\eta \in S^{n-1}$.

Proof. First of all for $x \neq y$ we have

$$G_x(x,y) = c_n \frac{(n-2)(x-y)}{|x-y|^n} - c_n \frac{(n-2)(|y|^2x-y)}{|\,x|y|-y/|y|\,|^n}.$$

Thus we have

$$\lim_{x\to\eta} G_x(x,y) = \frac{1}{\Omega_{n-1}} \frac{\eta(1-|y|^2)}{|\eta-y|^n}. \tag{2.6}$$

Let

$$G_1(x,y) := \frac{1}{\Omega_{n-1}} \frac{x-y}{|x-y|^n}, \tag{2.7}$$

and let

$$G_2(x,y) := \frac{1}{\Omega_{n-1}} \frac{y - |y|^2 x}{|\,x|y| - y/|y|\,|^n}. \tag{2.8}$$

The function G_2 is harmonic for $x \in B^n$. According to Lemma 2.5 it follows

$$\begin{aligned} \nabla G[f](x)h &= \int_{B^n} \langle G_x(x,y), h\rangle \; g(y)\, dy \\ &= \int_{B^n} \langle G_1(x,y), h\rangle \; g(y) dy + \int_{B^n} \langle G_2(x,y), h\rangle \; g(y) dy. \end{aligned} \tag{2.9}$$

The last statement of the lemma follows from relations (2.6) and (2.9) and Lebesgue Dominated Convergence Theorem.

□

Lemma 2.7. *Let u be a solution of the PDE $\Delta u = g$ ($g \in C(\overline{B^n})$) that maps the unit ball onto itself properly ($|u(x)| \to 1$ as $|x| \to 1$). Let in addition u be Lipschitz continuous. Then there exist almost everywhere in S^{n-1}:*

$$\nabla u(t) := \lim_{r\to 1-} \nabla u(rt) \tag{2.10}$$

and

$$J_u(t) := \lim_{r\to 1-} J_u(rt), \tag{2.11}$$

$t \in S^{n-1}$, and there holds the following relation:

$$\begin{aligned} J_u(t) &= \frac{D_\chi}{D_T} \int_{S^{n-1}} \frac{|f(t) - f(\eta)|^2}{|\eta - t|^n} d\sigma(\eta) \\ &+ \frac{D_\chi}{D_T} \int_0^1 r^{n-1} \Big(\int_{S^{n-1}} P(r\eta, t) \, \langle g(rt), f(\eta)\rangle \, d\sigma(\eta) \Big) dr, \;\; t \in S^{n-1}. \end{aligned} \tag{2.12}$$

Where D_χ and D_T are the square roots of Gram determinants of $\nabla\chi$ and ∇T, respectively.

If u is biharmonic ($\Delta\Delta u = 0$), then there holds:

$$\begin{aligned} J_u(t) &= \frac{D_\chi}{D_T} \int_{S^{n-1}} \frac{|f(t) - f(\eta)|^2}{|\eta - t|^n} d\sigma(\eta) \\ &+ \frac{D_\chi}{D_T} \int_0^1 r^{n-1} \left\langle g(r^2 t), f(t)\right\rangle dr, \;\; t \in S^{n-1}. \end{aligned} \tag{2.13}$$

For arbitrary continuous g and $|g| = \max_{|x|\le 1} |g(x)|$ there holds the inequality:

$$|J_u(t) - \frac{D_\chi}{D_T} \int_{S^{n-1}} \frac{|f(t) - f(\eta)|^2}{|\eta - t|^n} d\sigma(\eta)| \le \frac{D_\chi}{D_T} \frac{|g|}{n}, \;\; t \in S^{n-1}. \tag{2.14}$$

Proof. First of all, according to Lemma 2.6, $G[g]$ has a bounded derivative, and there exists the function $\nabla G[g](\eta)$, $\eta \in S^{n-1}$ which is continuous and satisfies the limit relation $\lim_{x\to\eta} \nabla G[g](x) = \nabla G[g](\eta)$. Since $u = P[f] - G[g]$ has bounded derivative, according to the Lemma 2.1 it follows that, there exists $\lim_{r\to 1-} \nabla P[f](r\eta) = \nabla P[f](\eta)$. Thus $\lim_{r\to 1-} \nabla u(r\eta) = \nabla u(\eta)$.

It follows that the mapping χ: $\chi(\theta) = f(T(\theta)) := f(t)$, $t \in S^{n-1}$, defines the outer normal vector field $\mathbf{n}_\chi$ almost everywhere in S^{n-1} at the point $\chi(\theta) = f(T(\theta)) = (\chi_1, \chi_2, \dots, \chi_n)$ by the formula:

$$\mathbf{n}_\chi(\chi(\theta)) = \chi_{\theta_1} \times \cdots \times \chi_{\theta_{n-2}} \times \chi_\varphi. \tag{2.15}$$

Since $\mathbf{n}_\chi(\chi(\theta))$ is the normal vector to the unit sphere, there holds the equality:

$$\mathbf{n}_\chi(\chi(\theta)) = D_\chi \cdot f(T(\theta)). \tag{2.16}$$

Let $u(S(r,\theta)) = (y_1, y_1, \dots, y_n)$, where S are spherical coordinates. According to Lemma 2.1, we obtain:

$$\lim_{r\to 1-} y_{i\varphi}(r,\theta) = \chi_{i\varphi}(\theta),\ i \in \{1,\dots,n\}, \tag{2.17}$$

$$\lim_{r\to 1-} y_{i\theta_j}(r,\theta) = \chi_{i\theta_j}(\theta),\ i \in \{1,\dots,n\},\ j \in \{1,\dots,n-2\}. \tag{2.18}$$

On the other hand, for almost every $\theta \in S^{n-1}$ we have

$$\frac{\chi_i(\theta) - y_i(r,\theta)}{1-r} = y_{ir}(\rho_{r,\theta},\theta)$$

where $r < \rho_{r,\theta} < 1$. Thus we have:

$$\lim_{r\to 1-} y_{ir}(r,\theta) = \lim_{r\to 1-} \frac{\chi_i(\theta) - y_i(r,\theta)}{1-r},\ i \in \{1,\dots,n\}. \tag{2.19}$$

Hence we derive

$$\begin{aligned}
\lim_{r\to 1-} J_{u\circ S}(r,\theta) &= \lim_{r\to 1-} \left\langle \frac{\chi - P[f]}{1-r}, \chi_{\theta_1} \times \cdots \times \chi_{\theta_{n-2}} \times \chi_\varphi \right\rangle + \Lambda \\
&= \lim_{r\to 1-} \int_{S^{n-1}} \frac{1+r}{|\eta - rt|^n} \left\langle \chi - f(\eta), \chi_{\theta_1} \times \cdots \times \chi_{\theta_{n-2}} \times \chi_\varphi \right\rangle d\sigma(\eta) + \Lambda \\
&= \lim_{r\to 1-} \int_{S^{n-1}} \frac{1+r}{|\eta - S(r,\theta)|^n} \langle f(T(\theta)) - f(\eta), \mathbf{n}_{f\circ T}(T(\theta))\rangle d\sigma(\eta) + \Lambda \\
&= \lim_{r\to 1-} D_\chi(\theta) \int_{S^{n-1}} \frac{1+r}{|\eta - S(r,\theta)|^n} \langle f(T(\theta)) - f(\eta), f(T(\theta))\rangle d\sigma(\eta) + \Lambda \\
&= \lim_{r\to 1-} \frac{1+r}{2} D_\chi(\theta) \int_{S^{n-1}} \frac{|f(T(\theta)) - f(\eta)|^2}{|\eta - S(r,\theta)|^n} d\sigma(\eta) + \Lambda.
\end{aligned} \tag{2.20}$$

Where $\Lambda = \lim_{r\to 1-} \left\langle \frac{G[g]}{1-r}, \chi_{\theta_1} \times \cdots \times \chi_{\theta_{n-2}} \times \chi_\varphi \right\rangle$.

In order to estimate Λ, observe first that:

$$G(x,y) = c_n \frac{|\,x|y| - y/|y|\,|^{n-2} - |x-y|^{n-2}}{|x-y|^{n-2} \cdot |\,x|y| - y/|y|\,|^{n-2}}. \tag{2.21}$$

Next

$$\begin{aligned}
&|\,x|y| - y/|y|\,|^{n-2} - |x-y|^{n-2} \\
&\quad = (|\,x|y| - y/|y|\,| - |x-y|) \sum_{k=1}^{n-2} (|\,x|y| - y/|y|\,|)^{n-2-k} \cdot |x-y|^{n-2-k},
\end{aligned} \tag{2.22}$$

and

$$\begin{aligned} |\,x|y| - y/|y|\,| - |x-y| &= \frac{|\,x|y| - y/|y|\,|^2 - |x-y|^2}{|\,x|y| - y/|y|\,| + |x-y|} \\ &= \frac{(1 + |x|^2|y|^2 - 2\langle x, y\rangle) - (|x|^2 + |y|^2 - 2\langle x, y\rangle)}{|\,x|y| - y/|y|\,| + |x-y|} \\ &= \frac{(1-|x|^2)(1-|y|^2)}{|\,x|y| - y/|y|\,| + |x-y|}. \end{aligned} \tag{2.23}$$

Inserting (2.22) and (2.23) into (2.21) we obtain:

$$\lim_{x\to t} \frac{G(x,y)}{1-|x|} = \frac{1}{\Omega_{n-1}} P(y,t). \tag{2.24}$$

On the other hand we have

$$\frac{1}{\Omega_{n-1}} \int_{B^n} P(y,t) \langle g(y), f(t)\rangle \, dy = \int_0^1 r^{n-1} \Big(\int_{S^{n-1}} P(r\eta, t) \langle g(r\eta), f(t)\rangle \, d\sigma(\eta)\Big) dr. \tag{2.25}$$

Next, there holds

$$J_{u\circ S}(r,\theta) = r^{n-1} J_u(rT(\theta)) \cdot D_T(\theta). \tag{2.26}$$

Combining (2.20), (2.24), (2.25) and (2.26) we obtain (2.12). Relations (2.13) and (2.14) follow form (2.12) and (1.2). If u is biharmonic, then g is harmonic and thus

$$\int_{S^{n-1}} P(r\eta, t) \langle g(r\eta), f(t)\rangle \, d\sigma(\eta) = \left\langle g(r^2 t), f(t)\right\rangle.$$

This yields the relation (2.13). □

Assume A is an $n \times n$ matrix with entries from $\mathbb{R}$. Define the (i,j)-minor $M_{i,j}$ of A as the determinant of the $(n-1)\times(n-1)$ matrix that results from deleting row i and column j of A, and the i,j cofactor of A as

$$C_{ij} = (-1)^{i+j} M_{ij}\,.$$

Then the adjugate of A is the $n \times n$ matrix

$$\tilde{A} = (C_{ji})_{i,j=1}^n\,.$$

I if A is an invertible matrix then

$$A^{-1} = \det(A)^{-1}\,\tilde{A}.$$

That is, the adjugate of A is the transpose of the "cofactor matrix" $(C_{ij})_{i,j=1}^n$ of A.

Lemma 2.8. *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator such that $A = [a_{ij}]_{i,j=1,\dots,n}$. If A is K quasiconformal, then there hold the following double inequality*

$$K^{1-n}|A|^{n-1}|x_1\times\cdots\times x_{n-1}| \le |Ax_1\times\cdots\times Ax_{n-1}| \le |A|^{n-1}|x_1\times\cdots\times x_{n-1}|. \tag{2.27}$$

Both inequalities in (2.27) are sharp.

Proof. Let $x_i = \sum_{j=1}^n x_{ij} e_j$, $i = 1, \dots n-1$. Then

$$Ax_1 \times \cdots \times Ax_{n-1} = \sum_\sigma \varepsilon_\sigma x_{1,\sigma_1} \dots x_{n-1\sigma_{n-1}} Ae_1 \times \cdots \times Ae_{n-1}.$$

It follows that

$$Ax_1 \times \cdots \times Ax_{n-1} = \tilde{A}x_1 \times \cdots \times x_{n-1}. \tag{2.28}$$

As A is K quasiconformal, $\tilde{A}$ is quasiconformal as well. Namely from $\tilde{A} = \det A \cdot A^{-1}$, it follows that

$$\tilde{\lambda}_k = \det A \cdot \frac{1}{\lambda_k}, \text{and } \tilde{\lambda}_n \le \tilde{\lambda}_{n-1} \le \cdots \le \tilde{\lambda}_1$$

and consequently

$$\frac{\tilde{\lambda}_1}{\tilde{\lambda}_n} \le K.$$

From (2.28) we obtain

$$|Ax_1 \times \cdots \times Ax_{n-1}| \le |\tilde{A}| \cdot |x_1 \times \cdots \times x_{n-1}|. \tag{2.29}$$

Furthermore

$$|\tilde{A}| = \tilde{\lambda}_1 = \frac{\det A}{\lambda_1} = \prod_{k=2}^n \lambda_k \le \lambda_n^{n-1} = |A|^{n-1}. \tag{2.30}$$

Equations (2.29) and (2.30) yields the right inequality of (2.27).

To obtain the left inequality of (2.27) we make use of (2.28) again. From (2.28) it follows that

$$|Ax_1 \times \cdots \times Ax_{n-1}| \ge \tilde{\lambda}_n \cdot |x_1 \times \cdots \times x_{n-1}|. \tag{2.31}$$

On the other hand

$$\tilde{\lambda}_n = \frac{\det A}{\lambda_n} = \prod_{k=1}^{n-1} \lambda_k \ge K^{n-1} \lambda_n^{n-1}.$$

This inequality completes the proof of lemma.

□

Lemma 2.9. *Let u be a solution of the PDE $\Delta u = g$, $g \in C(\overline{B}^n)$, that is Lipschitz continuous. Denote by ∇u its extension up to the boundary $S^{n-1} = \partial B^n$, which exists almost everywhere in S^{n-1}. Then for $x \in B^n$*

$$|\nabla u(x)| \le \text{ ess sup}_{|\eta|=1} |\nabla u(\eta)| + \left(1 + \frac{2\Gamma(\frac{n}{2})}{(n-1)\sqrt{\pi}\Gamma(\frac{n-1}{2})} \Omega_{n-1}\right) |g|, \tag{2.32}$$

where $|\cdot|$ is any norm of matrices and $|g| = \max\{|g(x)|, x \in \overline{B}^n\}$.

Proof. By taking the notation of Lemma 2.6 we have

$$\begin{aligned}\nabla u &= \nabla P[f](x) - \nabla G[g](x) \\ &= \nabla P[f](x) - \int_{B^n} G_1(x,y) g(y) dy - \int_{B^n} G_2(x,y) g(y) dy.\end{aligned}$$

Thus

$$\nabla u(x) + \int_{B^n} G_1(x,y) g(y) dy = \nabla P[f](x) - \int_{B^n} G_2(x,y) g(y) dy =: h(x).$$

Applying Lemma 2.2 to the harmonic mapping h, we have

$$|\nabla u(x)+\int_{B^n} G_1(x,y)g(y)dy| \le \text{ ess sup}_{|t|=1}|h(t)|$$
$$\le \text{ ess sup}_{|t|=1}|\nabla u(t)| + \sup_{|t|=1}|\int_{B^n} G_1(t,y)g(y)dy|.$$

Hence for $x\in B^n$ we have

$$|\nabla u(x)| \le \text{ess} \sup_{|t|=1}|\nabla u(t)| + \text{ ess sup}_{|x|\le 1}|\int_{B^n} G_1(x,y)g(y)dy|$$
$$+\,\text{ess sup}_{|t|=1}\int_{B^n} |G_1(t,y)||g(y)|dy.$$

Using now Lemma 2.3 we have

$$|\nabla u(x)| \le \text{ ess sup}_{|t|=1}|\nabla u(t)| + (1+\frac{2\Gamma(\frac{n}{2})}{(n-1)\sqrt{\pi}\Gamma(\frac{n-1}{2})}\Omega_{n-1})|g|.$$

□

Remark 2.10. It is known that harmonic and subharmonic functions satisfy the maximum principle. However, if $u\in C^2(B^n)\cap C^1(\overline{B^n})$ satisfies the PDE $\Delta u = g$, with

$$g\in C^1(\Omega),\ \ \langle \nabla u, \nabla g\rangle \le \frac{|g|^2}{n}, \tag{2.33}$$

then the mapping ∇u satisfies the maximum principle

$$\sup_{B^n}|\nabla u(x)| = \sup_{S^{n-1}}|\nabla u(x)|. \tag{2.34}$$

This estimate is better than the estimate (2.32), but the condition (2.33) is an essential one. For the details see [8, Theorem 15.1].

Lemma 2.11. *If $x\ge 0$ is a solution of the inequality $x\le ax^\alpha + b$, where $a\ge 1$ and $0\le a\alpha<1$, then*

$$x \le \frac{a+b-\alpha a}{1-\alpha a}. \tag{2.35}$$

Observe that for $\alpha=0$ (2.35) coincides with $x\le a+b$ i.e. $x\le ax^\alpha+b$.

Proof. We will use the Bernoulli's inequality. $x\le ax^\alpha+b = a(1+x-1)^\alpha + b \le a(1+\alpha(x-1))+b$. The relation (2.35) now easily follows. □

3. The main results

Theorem 3.1. *Let $K\ge 1$ be arbitrary and $n\in\mathbb{N}$ and let $g\in C(\overline{B^n})$. Then there exists a constant $M'=M'(n,K)$ such that:*

if u is K quasiconformal selfmapping of the unit ball B^n satisfying the PDE $\Delta u = g$, with $u(0)=0$ then:

$$|u(x)-u(y)|\le M'|x-y|,\ \ x,y\in B^n, \tag{3.1}$$

where $M' = M_1'(K,n)+M_2'(K,n)|g|$. Moreover if u is harmonic then $M'(n,K)\to 1$ as $K\to 1$.

Proof. Let $u(S(r,\theta)) = (y_1, y_1, \dots, y_n)$, where S are the spherical coordinates. Combining the Proposition 1.2 and Lemma 2.7, in the special case where the co-domain is the unit ball, we obtain that there exists ∇u, and J_u almost everywhere in S^{n-1} and there holds the following inequality:

$$J_u(t) \le \frac{D_\chi}{D_T}\left(\int_{S^{n-1}} \frac{|f(t)-f(\eta)|^2}{|\eta - t|^n} d\sigma(\eta) + \frac{|g|}{n}\right), \; t \in S^{n-1}. \tag{3.2}$$

Now from

$$|\nabla u(S(r,\theta))|^n \le K J_u(S(r,\theta)),$$

we obtain

$$\lim_{r\to 1-} |\nabla u(S(r,\theta))|^n \le \lim_{r\to 1-} K J_u(S(r,\theta)), \tag{3.3}$$

almost everywhere in K^{n-1}. From Lemma 2.1, we deduce that

$$\lim_{r\to 1-} \frac{\partial u\circ S}{\partial\theta_1}(r,\theta) \times \dots \times \frac{\partial u\circ S}{\partial\theta_{n-2}}(r,\theta) \times \frac{\partial u\circ S}{\partial\varphi}(r,\theta)$$

$$= \frac{\partial f\circ T}{\partial\theta_1}(\theta) \times \dots \times \frac{\partial f\circ T}{\partial\theta_{n-2}}(\theta) \times \frac{\partial f\circ T}{\partial\varphi}(\theta)$$

almost everywhere in K^{n-1}. Since

$$\frac{\partial u\circ S}{\partial\theta_i}(r,\theta) = r u'(S(r,\theta))\frac{\partial T}{\partial\theta_i},$$

using (2.27) we obtain that

$$D_\chi(\theta) \le \lim_{r\to 1-} |\nabla u(S(r,\theta))|^{n-1} D_T(\theta). \tag{3.4}$$

From (3.2)-(3.4) we infer that

$$|\nabla u(T(\theta))|^n \le K|\nabla u(T(\theta)|^{n-1}\left(\int_{S^{n-1}} \frac{|f(T(\theta))-f(\eta)|^2}{|\eta - T(\theta)|^n} d\sigma(\eta) + \frac{|g|}{n}\right)$$

i.e.

$$|\nabla u(T(\theta))| \le K\left(\int_{S^{n-1}} \frac{|f(T(\theta))-f(\eta)|^2}{|\eta - T(\theta)|^n} d\sigma(\eta) + \frac{|g|}{n}\right). \tag{3.5}$$

In view of Lemma 2.9, for every $\varepsilon > 0$ there exists $\theta_\varepsilon \in K^{n-1}$ such that:

$$\begin{aligned} M :&= \operatorname{ess\,sup}\{|\nabla u(x)| : |x| < 1\} \\ &\le (1-\varepsilon)^{-1}\left(|\nabla u(T(\theta_\varepsilon))| + (1 + \frac{2\Gamma(\frac{n}{2})}{(n-1)\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}\Omega_{n-1})|g|\right). \end{aligned} \tag{3.6}$$

The mean value theorem yields

$$|u(x) - u(y)| \le \sup_{t\in B^n} |\nabla u(t)| \cdot |x-y|. \tag{3.7}$$

Let $\mu = K^{1/(1-n)}$. It is clear that $0 < \mu \le 1$. Let $\gamma = 1 - n + \mu^2$, and let $\nu = 1-\mu$. Now applying the relation (3.5) for $\theta = \theta_\varepsilon$, and using (1.4), (3.6) and (3.7), we obtain

$$\begin{aligned}&(1-\varepsilon)M-(1+\frac{2\Gamma(\frac{n}{2})}{(n-1)\sqrt{\pi}\Gamma(\frac{n-1}{2})}\Omega_{n-1})|g|\\&\le K\left((Ml)^{\nu}\int_{S^{n-1}}|\eta-T(\theta_\varepsilon)|^{\gamma}\frac{|f(T(\theta_\varepsilon))-f(\eta)|^{2-\nu}}{|T(\theta_\varepsilon)-\eta|^{\mu^2+\mu}}d\sigma(\eta)+\frac{|g|}{n}\right)\\&\le K(Ml)^{\nu}M_1(K,n)^{1+\mu}\int_{S^{n-1}}|\eta-T(\theta_\varepsilon)|^{\gamma}d\sigma(\eta)+K\frac{|g|}{n}.\end{aligned}$$

Letting $\varepsilon\to 0$ we obtain

$$M\le M_2(K,n)M^{\nu}+M_3(K,n)|g|,$$

where

$$M_2(K,n)=Kl^{\nu}M_1(K,n)^{1+\mu}\int_{S^{n-1}}|\eta-T(\theta_0)|^{\gamma}d\sigma(\eta)$$

and

$$M_3(K,n)=(1+\frac{2\Gamma(\frac{n}{2})}{(n-1)\sqrt{\pi}\Gamma(\frac{n-1}{2})}\Omega_{n-1})+\frac{K}{n}.$$

First of all, there holds

$$M\le M_4:=(M_2(K)+M_3(K,n)|g|)^{1/(1-\nu)}=(M_2(K)+M_3(K,n)|g|)^{K}. \tag{3.8}$$

If $\nu M_2(K)<1$, from Lemma 2.11 we obtain

$$M\le M_5:=\frac{M_2(K,n)+M_3(K,n)|g|-\nu M_2(K,n)}{1-\nu M_2(K,n)}. \tag{3.9}$$

Therefore the inequality (3.1) does hold for

$$M'=\min(\{M_4\}\cup\{M_5:\nu M_2(K)<1\}).$$

Using (1.4), Lemma 2.8 and Lemma 2.4, it follows that $\lim_{K\to 1}M'(K,n)=1$ if $g=0$. □

Concerning the co-Lipschitz character of these mappings we have the following partial result.

Theorem 3.2. *Let $K<2^{n-1}$ and assume that u is a $K-$q.c. solution of PDE $\Delta u=g$ that maps the unit ball onto itself satisfying the following conditions:*
i) $u\in C^1(\overline{B^n})$,
ii) $g\in C(\overline{B^n})$ such that $|g|_\infty<M_0(K,n)$ where $M_0(K,n)$ is given in (3.14).
Then u is co-Lipschitz.

Proof. From (2.14) we obtain

$$J_u(t)\ge\frac{D_\chi}{D_T}\int_{S^{n-1}}\frac{|f(t)-f(\eta)|^2}{|\eta-t|^n}d\sigma(\eta)-\frac{D_\chi}{D_T}\frac{|g|}{n},\ \ t\in S^{n-1}. \tag{3.10}$$

Using (2.27) we obtain

$$K^{1-n}\lim_{r\to 1-}|\nabla u(S(r,\theta))|^{n-1}\le\frac{D_\chi}{D_T}\le\lim_{r\to 1-}|\nabla u(S(r,\theta))|^{n-1}. \tag{3.11}$$

Combining (3.10) and (3.11) it follows that

$$\lim_{r\to 1-}|\nabla u(S(r,\theta))|^n \geq K^{1-n}|\lim_{r\to 1-}|\nabla u(S(r,\theta))|^{n-1}\int_{S^{n-1}}\frac{|f(t)-f(\eta)|^2}{|\eta-t|^n}d\sigma(\eta)$$
$$-\lim_{r\to 1-}|\nabla u(S(r,\theta))|^{n-1}\frac{|g|}{n},\ \ t\in S^{n-1},$$

i.e.

$$\lim_{r\to 1-}|\nabla u(S(r,\theta))| \geq K^{1-n}\int_{S^{n-1}}\frac{|f(t)-f(\eta)|^2}{|\eta-t|^n}d\sigma(\eta)-\frac{|g|}{n},\ \ t\in S^{n-1}. \tag{3.12}$$

As u^{-1} is K-q.c, using (1.4) and (3.12) one gets

$$\lim_{r\to 1-}|\nabla u(S(r,\theta))| \geq \frac{M_0(K,n)-|g|}{n},\ \ t\in S^{n-1}, \tag{3.13}$$

where

$$M_0(K,n)=\int_{S^{n-1}}\frac{nK^{1-n}(M_1(K,n))^{2K^{1/(1-n)}}}{|\eta-t|^{n-2K^{1/(n-1)}}}d\sigma(\eta). \tag{3.14}$$

The rest of the proof follows from the condition i) and [18, Lemma 4.5].

□

3.1. **Examples of q.c. mappings satisfying PDE $\Delta u = g$.** In the following example (a), it is shown that for the class of radial twice differentiable q.c. selfmapping of the unit ball (which is quite large), Theorem 3.1 yields also a sufficient condition. In its particular case (b) is shown that the condition $K < 2^n$ of Theorem 3.2, is the best possible.

a) Define $u(x) = h(|x|)x$, where $r \mapsto rh(r)$ is a twice differentiable diffeomorphism of $[0, 1)$ onto itself. Then, for $r = |x|$,

$$J_u(x)=h^n(r)\left(1+\frac{h'(r)}{h(r)}r\right), \tag{3.15}$$

and

$$|\nabla u(x)|^n=h^n(r)\left(1+\frac{h'(r)}{h(r)}r\right)^n. \tag{3.16}$$

From (3.15) and (3.16) we obtain

$$\frac{|\nabla u(x)|^n}{J_u(x)}=\left(1+\frac{h'(r)}{h(r)}r\right)^{n-1}.$$

Thus u is a selfmapping of the unit ball satisfying PDE

$$\Delta u(x)=g(x):=\left(h''(r)+\frac{(n+1)h'(r)}{r}\right)x,$$

and it is quasiconformal if and only if

$$\limsup_{r\to 1}h'(r)<\infty, \tag{3.17}$$

or what is the same if and only if $|\nabla u(x)|$ is bounded.

b) In particular, take $u(x) = |x|^\alpha x$, with $\alpha \geq 1$. Then

$$J_u(x)=(1+\alpha)|x|^{n\alpha}, \tag{3.18}$$

and

$$|\nabla u(x)| = (\alpha+1)|x|^\alpha. \tag{3.19}$$

By (3.18) and (3.19) it follows that

$$\frac{|\nabla u(x)|^n}{J_u(x)} = (\alpha+1)^{n-1}.$$

Therefore u is twice differentiable $(1+\alpha)^{n-1}$-quasiconformal self-mapping of the unit ball with $J_u(0) = 0$. This means that the constant 2^{n-1} is the best possible.

3.2. **Remarks.** It is well known that the harmonic extension (via Poisson integral) of a homeomorphism of the unit circle is always a diffeomorphism of the unit disk. In higher dimensions, however, the situation is quite different. Namely Melas ([22] see also [16])) constructed a homeomorphism of the unit sphere S^{n-1} $(n \geq 3)$ whose harmonic extension fails to be diffeomorphic. The questions arises, do there exist such examples, assuming both conditions, harmonicity and quasiconformality; in other words do some q.c. harmonic mappings have critical points i.e. the points in which the Jacobian is zero? It seems that for $K \leq 2^{n-1}$, such example do not exists. In [19] and [3] is treated this problem in the complex plane. For this problem concerning, hyperbolic harmonic mappings between surfaces see [29] and [17], and for q.c. hyperbolic harmonic selfmapping of the unit ball see [18].

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